Heisenberg’s uncertainty principle

**Lemma**

If

\[ \hat{x}\psi = i \hbar \hat{p}\psi \]  

(6.29)
Heisenberg’s uncertainty principle

Lemma

If

$$\hat{x}\psi = ia\hat{p}\psi$$

for some real parameter $a$, then $$(\Delta_\psi x)(\Delta_\psi p) = \frac{1}{2} \hbar.$$
Heisenberg’s uncertainty principle

Lemma

If

\[ \hat{x}\psi = ia\hat{p}\psi \]  \hspace{1cm} (6.29)

for some real parameter \( a \), then \( (\Delta_\psi x)(\Delta_\psi p) = \frac{1}{2}\hbar \).

Proof

If \( \hat{x}\psi = ia\hat{p}\psi \) then we have
Heisenberg’s uncertainty principle

**Lemma**

If

\[ \hat{x}\psi = ia\hat{p}\psi \]  \hspace{1cm} (6.29)

for some real parameter \( a \), then \((\Delta_{\psi} x)(\Delta_{\psi} p) = \frac{1}{2}\hbar\).

**Proof**

If \( \hat{x}\psi = ia\hat{p}\psi \) then we have

\[ (\psi, \{\hat{x}, \hat{p}\}\psi) = (\psi, \hat{x}\hat{p}\psi) + (\psi, \hat{p}\hat{x}\psi) \]
Heisenberg’s uncertainty principle

Lemma

If

\[ \hat{x}\psi = ia\hat{p}\psi \]  \hspace{1cm} (6.29)

for some real parameter \( a \), then \((\Delta_\psi x)(\Delta_\psi p) = \frac{1}{2}\hbar\).

Proof

If \( \hat{x}\psi = ia\hat{p}\psi \) then we have

\[
(\psi, \{\hat{x}, \hat{p}\}\psi) = (\psi, \hat{x}\hat{p}\psi) + (\psi, \hat{p}\hat{x}\psi)
\]

\[
= (\hat{x}\psi, \hat{p}\psi) + (\hat{p}\psi, \hat{x}\psi)
\]
**Lemma**

If

\[ \hat{x}\psi = ia\hat{p}\psi \]  \hspace{1cm} (6.29)

for some real parameter \( a \), then

\[ (\Delta\psi x)(\Delta\psi p) = \frac{1}{2}\hbar. \]

**Proof**  
If \( \hat{x}\psi = ia\hat{p}\psi \) then we have

\[
(\psi, \{\hat{x}, \hat{p}\}\psi) = (\psi, \hat{x}\hat{p}\psi) + (\psi, \hat{p}\hat{x}\psi) \\
= (\hat{x}\psi, \hat{p}\psi) + (\hat{p}\psi, \hat{x}\psi) \\
= (ia\hat{p}\psi, \hat{p}\psi) + (\hat{p}\psi, ia\hat{p}\psi)
\]
Heisenberg’s uncertainty principle

**Lemma**

If

\[
\hat{x}\psi = ia\hat{p}\psi
\]  \hspace{1cm} (6.29)

for some real parameter \(a\), then \((\Delta_{\psi x})(\Delta_{\psi p}) = \frac{1}{2}\hbar\).

**Proof**

If \(\hat{x}\psi = ia\hat{p}\psi\) then we have

\[
(\psi, \{\hat{x}, \hat{p}\}\psi) = (\psi, \hat{x}\hat{p}\psi) + (\psi, \hat{p}\hat{x}\psi)
= (\hat{x}\psi, \hat{p}\psi) + (\hat{p}\psi, \hat{x}\psi)
= (ia\hat{p}\psi, \hat{p}\psi) + (\hat{p}\psi, ia\hat{p}\psi)
= (-ia + ia)(\hat{p}\psi, \hat{p}\psi) = 0,
\]
Heisenberg’s uncertainty principle

Lemma

If

\[ \hat{x}\psi = ia\hat{p}\psi \]  \hspace{1cm} (6.29)

for some real parameter \( a \), then \( (\Delta_x \psi)(\Delta_p \psi) = \frac{1}{2}\hbar \).

Proof

If \( \hat{x}\psi = ia\hat{p}\psi \) then we have

\[
\begin{align*}
(\psi, \{\hat{x}, \hat{p}\}\psi) &= (\psi, \hat{x}\hat{p}\psi) + (\psi, \hat{p}\hat{x}\psi) \\
&= (\hat{x}\psi, \hat{p}\psi) + (\hat{p}\psi, \hat{x}\psi) \\
&= (ia\hat{p}\psi, \hat{p}\psi) + (\hat{p}\psi, ia\hat{p}\psi) \\
&= (-ia + ia)(\hat{p}\psi, \hat{p}\psi) = 0,
\end{align*}
\]

This is the condition for the first term on the RHS of (6.26) to vanish.
Heisenberg’s uncertainty principle

Lemma

If

\[ \hat{x}\psi = ia\hat{p}\psi \quad (6.29) \]

for some real parameter a, then \((\Delta_\psi x)(\Delta_\psi p) = \frac{1}{2}\hbar.\)

Proof

If \(\hat{x}\psi = ia\hat{p}\psi\) then we have

\[
(\psi, \{\hat{x}, \hat{p}\}\psi) = (\psi, \hat{x}\hat{p}\psi) + (\psi, \hat{p}\hat{x}\psi) \\
= (\hat{x}\psi, \hat{p}\psi) + (\hat{p}\psi, \hat{x}\psi) \\
= (ia\hat{p}\psi, \hat{p}\psi) + (\hat{p}\psi, ia\hat{p}\psi) \\
= (-ia + ia)(\hat{p}\psi, \hat{p}\psi) = 0, \\
\]

This is the condition for the first term on the RHS of (6.26) to vanish. We also have that \(\langle \hat{x} \rangle_\psi = ia\langle \hat{p} \rangle_\psi\)
Heisenberg’s uncertainty principle

Lemma

If

\[ \hat{x}\psi = ia\hat{p}\psi \quad (6.29) \]

for some real parameter \( a \), then \( (\Delta_\psi x)(\Delta_\psi p) = \frac{1}{2}\hbar \).

Proof

If \( \hat{x}\psi = ia\hat{p}\psi \) then we have

\[
(\psi, \{\hat{x}, \hat{p}\}\psi) = (\psi, \hat{x}\hat{p}\psi) + (\psi, \hat{p}\hat{x}\psi) \\
= (\hat{x}\psi, \hat{p}\psi) + (\hat{p}\psi, \hat{x}\psi) \\
= (ia\hat{p}\psi, \hat{p}\psi) + (\hat{p}\psi, ia\hat{p}\psi) \\
= (-ia + ia)(\hat{p}\psi, \hat{p}\psi) = 0, 
\]

This is the condition for the first term on the RHS of (6.26) to vanish. We also have that \( \langle \hat{x} \rangle_\psi = ia\langle \hat{p} \rangle_\psi \) and, since both expectations are real, this implies that \( \langle \hat{x} \rangle_\psi = \langle \hat{p} \rangle_\psi = 0 \).
### Lemma

If

\[ \hat{x}\psi = i\hbar \hat{p}\psi \]  \hspace{1cm} (6.29)

for some real parameter \( a \), then \((\Delta \psi x)(\Delta \psi p) = \frac{1}{2} \hbar\).

#### Proof

If \( \hat{x}\psi = i\hbar \hat{p}\psi \) then we have

\[
\begin{align*}
(\psi, \{\hat{x}, \hat{p}\}\psi) &= (\psi, \hat{x}\hat{p}\psi) + (\psi, \hat{p}\hat{x}\psi) \\
&= (\hat{x}\psi, \hat{p}\psi) + (\hat{p}\psi, \hat{x}\psi) \\
&= (i\hbar\hat{p}\psi, \hat{p}\psi) + (\hat{p}\psi, i\hbar\hat{p}\psi) \\
&= (-ia + ia)(\hat{p}\psi, \hat{p}\psi) = 0,
\end{align*}
\]

This is the condition for the first term on the RHS of (6.26) to vanish. We also have that \( \langle \hat{x} \rangle_\psi = i\hbar \langle \hat{p} \rangle_\psi \) and, since both expectations are real, this implies that \( \langle \hat{x} \rangle_\psi = \langle \hat{p} \rangle_\psi = 0 \). Hence

\[
(\hat{x} - \langle \hat{x} \rangle_\psi)\psi = i\hbar(\hat{p} - \langle \hat{p} \rangle_\psi)\psi,
\]

from (6.29).
Heisenberg’s uncertainty principle

Lemma

If

\[ \hat{x}_\psi = i a \hat{p} \psi \]

for some real parameter \( a \), then \( (\Delta _{\psi} x)(\Delta _{\psi} p) = \frac{1}{2} \hbar \).

Proof

If \( \hat{x}_\psi = i a \hat{p} \psi \) then we have

\[ \langle \psi, \{ \hat{x}, \hat{p} \} \psi \rangle = \langle \psi, \hat{x} \hat{p} \psi \rangle + \langle \psi, \hat{p} \hat{x} \psi \rangle \]

\[ = \langle \hat{x}_\psi, \hat{p} \psi \rangle + \langle \hat{p}_\psi, \hat{x} \psi \rangle \]

\[ = \langle i a \hat{p} \psi, \hat{p} \psi \rangle + \langle \hat{p} \psi, i a \hat{p} \psi \rangle \]

\[ = (-ia + ia)(\hat{p} \psi, \hat{p} \psi) = 0, \]

This is the condition for the first term on the RHS of (6.26) to vanish. We also have that \( \langle \hat{x} \rangle_{\psi} = i a \langle \hat{p} \rangle_{\psi} \) and, since both expectations are real, this implies that \( \langle \hat{x} \rangle_{\psi} = \langle \hat{p} \rangle_{\psi} = 0 \). Hence

\[ (\hat{x} - \langle \hat{x} \rangle_{\psi}) \psi = i a (\hat{p} - \langle \hat{p} \rangle_{\psi}) \psi, \]

and we have equality in (6.24) and hence (6.28).
Heisenberg’s uncertainty principle

\[ \hat{x}\psi = i\hbar \hat{p}\psi \]  

(6.29)

**Lemma**

*The condition (6.29) holds if and only if \( \psi(x) = C \exp(-bx^2) \) for some positive constants \( b, C \).*

**Correction:** the condition (6.29) holds for any function of this form, regardless of whether \( b \) and \( C \) are positive. However, it only defines a normalisable wavefunction for positive \( b \) and nonzero \( C \).
Heisenberg's uncertainty principle

\[ \hat{x}\psi = i\hat{a}\hat{p}\psi \]  \hspace{1cm} (6.29)

Lemma

The condition (6.29) holds if and only if \( \psi(x) = C \exp(-bx^2) \) for some positive constants \( b, C \).

Proof.

If \( \hat{x}\psi = i\hat{a}\hat{p}\psi \) for some real \( a \),
Heisenberg’s uncertainty principle

\[ \hat{x}\psi = i\hbar \hat{p}\psi \quad (6.29) \]

**Lemma**

The condition (6.29) holds if and only if \( \psi(x) = C \exp(-bx^2) \) for some positive constants \( b, C \).

**Proof.**

If \( \hat{x}\psi = i\hbar \hat{p}\psi \) for some real \( a \), we have that \( x\psi = a\hbar \frac{\partial}{\partial x} \psi \).
Heisenberg's uncertainty principle

\[ \hat{x}\psi = i\hbar\hat{p}\psi \quad (6.29) \]

Lemma

The condition (6.29) holds if and only if \( \psi(x) = C \exp(-bx^2) \) for some positive constants \( b, C \).

Proof.

If \( \hat{x}\psi = i\hbar\hat{p}\psi \) for some real \( a \), we have that \( x\psi = a\hbar \frac{\partial}{\partial x} \psi \) and so \( \psi(x) = C \exp(-bx^2) \) for some real \( b = -\frac{a\hbar}{2} \).

⇒ minimum uncertainty.
Heisenberg’s uncertainty principle

\[ \hat{x}\psi = i\hbar \hat{p}\psi \]  \hspace{2cm} (6.29)

**Lemma**

The condition (6.29) holds if and only if \( \psi(x) = C \exp(-bx^2) \) for some positive constants \( b, C \).

**Proof.**

If \( \hat{x}\psi = i\hbar \hat{p}\psi \) for some real \( a \), we have that \( x\psi = a\hbar \frac{\partial}{\partial x} \psi \) and so \( \psi(x) = C \exp(-bx^2) \) for some real \( b = -\frac{a\hbar}{2} \), and because we have equality in (6.28) we know the uncertainty is minimised.
Heisenberg’s uncertainty principle

\[ \hat{x}\psi = i\hbar \hat{p}\psi \]  \hspace{1cm} (6.29)

**Lemma**

The condition (6.29) holds if and only if \( \psi(x) = C \exp(-bx^2) \) for some positive constants \( b, C \).

**Proof.**

If \( \hat{x}\psi = i\hbar \hat{p}\psi \) for some real \( a \), we have that \( x\psi = a\hbar \frac{\partial}{\partial x} \psi \) and so \( \psi(x) = C \exp(-bx^2) \) for some real \( b = -\frac{a\hbar}{2} \), and because we have equality in (6.28) we know the uncertainty is minimised. Conversely, any wavefunction of the form \( \psi(x) = C \exp(-bx^2) \) satisfies \( \hat{x}\psi = i\hbar \hat{p}\psi \) for some real \( a \).
Heisenberg’s uncertainty principle

\[ \hat{x}\psi = i\hbar \hat{p}\psi \]  
\[ (6.29) \]

Lemma

The condition (6.29) holds if and only if \( \psi(x) = C \exp(-bx^2) \) for some positive constants \( b, C \).

Proof.

If \( \hat{x}\psi = i\hbar \hat{p}\psi \) for some real \( a \), we have that \( x\psi = a\hbar \frac{\partial}{\partial x}\psi \) and so \( \psi(x) = C \exp(-bx^2) \) for some real \( b = -\frac{a\hbar}{2} \), and because we have equality in (6.28) we know the uncertainty is minimised.

Conversely, any wavefunction of the form \( \psi(x) = C \exp(-bx^2) \) satisfies \( \hat{x}\psi = i\hbar \hat{p}\psi \) for some real \( a \).

Gaussian wavepackets are the minimum uncertainty states with \( \langle x \rangle = \langle p \rangle = 0 \). With a bit more algebra one can generalise this to non-zero expectation values.
What does the uncertainty principle tell us?

The uncertainty principle is a mathematical statement relating the uncertainties of $x$ and $p$ which are quantities defined for a given state $\psi$. 
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What does the uncertainty principle tell us?

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Heisenberg originally suggested that the uncertainty principle can be understood simply by observing that a measurement of $A$ creates uncertainty by disturbing the value of any observable $B$ that does not commute with $A$. 
What does the uncertainty principle tell us?

The uncertainty principle is a mathematical statement relating the uncertainties of $x$ and $p$ which are quantities defined for a given state $\psi$. For example, for any state $\psi$ with $\Delta_{\psi} x = \delta$, we have $\Delta_{\psi} p \geq \frac{\hbar}{2\delta}$.

Heisenberg originally suggested that the uncertainty principle can be understood simply by observing that a measurement of $A$ creates uncertainty by disturbing the value of any observable $B$ that does not commute with $A$. This is not a valid argument!
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Problems with the naive interpretation of the uncertainty principle:
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Heisenberg originally suggested that the uncertainty principle can be understood simply by observing that a measurement of $A$ creates uncertainty by disturbing the value of any observable $B$ that does not commute with $A$. This is not a valid argument!

Problems with the naive interpretation of the uncertainty principle:

- There is generally no definite fixed pre-measurement value of either $A$ or $B$. (Not the case that an electron had definite momentum before we measured its position (nor afterwards).)
What does the uncertainty principle tell us?

The uncertainty principle is a mathematical statement relating the uncertainties of $x$ and $p$ which are quantities defined for a given state $\psi$. For example, for any state $\psi$ with $\Delta_{\psi} x = \delta$, we have $\Delta_{\psi} p \geq \frac{\hbar}{2\delta}$.

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Problems with the naive interpretation of the uncertainty principle:

- There is generally no definite fixed pre-measurement value of either $A$ or $B$.
- The mathematical derivation of the uncertainty principle does not require us to consider measurements of $A$ or $B$ actually taking place.
What does the uncertainty principle tell us?

The uncertainty principle is a mathematical statement relating the uncertainties of $x$ and $p$ which are quantities defined for a given state $\psi$. For example, for any state $\psi$ with $\Delta_\psi x = \delta$, we have $\Delta_\psi p \geq \frac{\hbar}{2\delta}$.

Heisenberg originally suggested that the uncertainty principle can be understood simply by observing that a measurement of $A$ creates uncertainty by disturbing the value of any observable $B$ that does not commute with $A$. This is not a valid argument!

Problems with the naive interpretation of the uncertainty principle:

- There is generally no definite fixed pre-measurement value of either $A$ or $B$.
- The mathematical derivation of the uncertainty principle does not require us to consider measurements of $A$ or $B$ actually taking place. The quantity $\Delta_\psi A$ is mathematically defined whether or not we choose to carry out a measurement of $A$. 
Ehrenfest’s theorem

The expectation value $\langle A \rangle_\psi$ of an operator $A$ in the state $\psi$ evolves by
Ehrenfest’s theorem

**Theorem**

The expectation value $\langle A \rangle_\psi$ of an operator $A$ in the state $\psi$ evolves by

$$\frac{d}{dt} \langle A \rangle_\psi$$
Ehrenfest’s theorem

**Theorem**

The expectation value $\langle A \rangle_\psi$ of an operator $A$ in the state $\psi$ evolves by

$$ \frac{d}{dt} \langle A \rangle_\psi = \frac{i}{\hbar} \langle [\hat{H}, A] \rangle_\psi + \langle \frac{\partial A}{\partial t} \rangle_\psi. \quad (6.30) $$
Ehrenfest’s theorem

Proof.

We have

\[
\frac{d\langle A \rangle_\psi}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} \psi^* A \psi \, dx
\]

definition of expectation value \( \langle A \rangle_\psi \)
Proof.

We have

\[
\frac{d\langle A\rangle_\psi}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} \psi^* A\psi \, dx
\]

by the product rule.

\[
\int_{-\infty}^{\infty} \left( \frac{\partial \psi^*}{\partial t} A\psi + \psi^* \frac{\partial A}{\partial t} \psi + \psi^* A \frac{\partial \psi}{\partial t} \right) dx
\]
Ehrenfest’s theorem

Proof.

We have

\[
\frac{d\langle A \rangle_\psi}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} \psi^* A \psi \, dx
\]

\[
= \int_{-\infty}^{\infty} \left( \frac{\partial \psi^*}{\partial t} A \psi + \psi^* \frac{\partial A}{\partial t} \psi + \psi^* A \frac{\partial \psi}{\partial t} \right) \, dx
\]

\[
= \langle \frac{\partial A}{\partial t} \rangle_\psi + \frac{i}{\hbar} \int_{-\infty}^{\infty} ((\hat{H} \psi)^* A \psi - \psi^* A (\hat{H} \psi)) \, dx
\]

\[
\hat{H} \psi = \hat{H} \psi
\]

\[
\hat{H} \psi^* = -\hat{H} \psi^*
\]
Proof.

We have

\[
\frac{d\langle A\rangle_\psi}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} \psi^* A\psi \, dx
\]

\[
= \int_{-\infty}^{\infty} \left( \frac{\partial \psi^*}{\partial t} A\psi + \psi^* \frac{\partial A}{\partial t} \psi + \psi^* A \frac{\partial \psi}{\partial t} \right) \, dx
\]

\[
= \langle \frac{\partial A}{\partial t} \rangle_\psi + i \hbar \int_{-\infty}^{\infty} \left( (\hat{H}\psi)^* A\psi - \psi^* A(\hat{H}\psi) \right) \, dx
\]

\[
= \langle \frac{\partial A}{\partial t} \rangle_\psi + i \hbar \int_{-\infty}^{\infty} \left( \psi^* \hat{H} A\psi - \psi^* A(\hat{H}\psi) \right) \, dx
\]
Ehrenfest’s theorem

Proof.

We have

\[
\frac{d\langle A\rangle_\psi}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} \psi^* A \psi \, dx
\]

\[
= \int_{-\infty}^{\infty} \left( \frac{\partial \psi^*}{\partial t} A \psi + \psi^* \frac{\partial A}{\partial t} \psi + \psi^* A \frac{\partial \psi}{\partial t} \right) \, dx
\]

\[
= \langle \frac{\partial A}{\partial t} \rangle_\psi + \frac{i}{\hbar} \int_{-\infty}^{\infty} \left( (\hat{H} \psi)^* A \psi - \psi^* A (\hat{H} \psi) \right) \, dx
\]

\[
= \langle \frac{\partial A}{\partial t} \rangle_\psi + \frac{i}{\hbar} \int_{-\infty}^{\infty} \left( \psi^* \hat{H} A \psi - \psi^* A (\hat{H} \psi) \right) \, dx
\]

\[
= \frac{i}{\hbar} \langle [\hat{H}, A] \rangle_\psi + \langle \frac{\partial A}{\partial t} \rangle_\psi.
\]

(6.31)
Applications of Ehrenfest’s theorem

For $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$, we have
Applications of Ehrenfest’s theorem

For $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$, we have

$$[\hat{H}, \hat{p}] = [V(x), \hat{p}]$$

$$\left[\frac{i}{\hbar} \frac{\hat{\partial}}{\partial x} \cdot V(x), \hat{p} \right]$$
For $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$, we have

\[
[\hat{H}, \hat{p}] = [V(x), \hat{p}]
\]

\[
= [V(x), -i\hbar \frac{\partial}{\partial x}]
\]
Applications of Ehrenfest’s theorem

For $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$, we have

$$[\hat{H}, \hat{p}] = [V(x), \hat{p}]$$

$$= [V(x), -i\hbar \frac{\partial}{\partial x}]$$

$$= i\hbar \frac{dV}{dx} \tag{6.32}$$

$$V(x) \left( -i\hbar \frac{\partial}{\partial x} \right) \psi = \left( -i\hbar \frac{\partial}{\partial x} \right) (V(x) \psi)$$

$$= V(x) \left( -i\hbar \frac{\partial \psi}{\partial x} \right) - V(x) \left( -i\hbar \frac{\partial \psi}{\partial x} \right) + i\hbar \frac{dV}{dx} \psi(x)$$
Applications of Ehrenfest’s theorem

For \( \hat{H} = \frac{\hat{p}^2}{2m} + V(x) \), we have

\[
[\hat{H}, \hat{p}] = [V(x), \hat{p}]
\]
\[
= [V(x), -i\hbar \frac{\partial}{\partial x}]
\]
\[
= i\hbar \frac{dV}{dx}
\]

(6.32)

\[
[\hat{H}, \hat{x}] = \left[ \frac{\hat{p}^2}{2m}, \hat{x} \right]
\]

\[
\]

\[
\hat{p} \hat{x} = \delta [\hat{p}, \hat{x}] + [\hat{p}, \hat{x}] \hat{p}
\]
Applications of Ehrenfest’s theorem

For $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$, we have

$$[\hat{H}, \hat{p}] = [V(x), \hat{p}] = [V(x), -i\hbar \frac{\partial}{\partial x}] = i\hbar \frac{dV}{dx}$$

(6.32)

$$[\hat{H}, \hat{x}] = \left[ \frac{\hat{p}^2}{2m}, \hat{x} \right] = \frac{1}{2m} 2[\hat{p}, \hat{x}]\hat{p} = -\frac{i\hbar}{m} \hat{p}$$

(6.33)
Applying to $\hat{p}$
\[
[\hat{H}, \hat{p}] = [V(x), \hat{p}] = [V(x), -i\hbar \frac{\partial}{\partial x}]
\]
\[= i\hbar \frac{dV}{dx} \tag{6.32}\]

Applying to $\hat{x}$
\[
[\hat{H}, \hat{x}] = \left[ \hat{\frac{\hat{p}^2}{2m}}, \hat{x} \right]
\]
\[= \frac{1}{2m} 2[\hat{p}, \hat{x}]\hat{p} = -\frac{i\hbar}{m} \hat{p} \tag{6.33}\]

Applying to $\hat{H}$
\[
[\hat{H}, \hat{H}] = 0. \tag{6.34}\]
Since none of these operators is explicitly time-dependent, we have that \[ \frac{\partial \hat{H}}{\partial t} = \frac{\partial \hat{x}}{\partial t} = \frac{\partial \hat{p}}{\partial t} = 0 \]
Applications of Ehrenfest’s theorem

Since none of these operators is explicitly time-dependent, we have that \( \frac{\partial \hat{H}}{\partial t} = \frac{\partial \hat{x}}{\partial t} = \frac{\partial \hat{p}}{\partial t} = 0 \) and so the \( \langle \frac{\partial A}{\partial t} \rangle_\psi \) term on the RHS of (6.30) vanishes in each case, giving

\[
\frac{d}{dt} \langle A \rangle_\psi = \frac{i}{\hbar} \langle [\hat{H}, A] \rangle_\psi + \langle \frac{\partial A}{\partial t} \rangle_\psi. \tag{6.30}
\]
Since none of these operators is explicitly time-dependent, we have that \( \frac{\partial \hat{H}}{\partial t} = \frac{\partial \hat{x}}{\partial t} = \frac{\partial \hat{p}}{\partial t} = 0 \) and so the \( \langle \frac{\partial A}{\partial t} \rangle_\psi \) term on the RHS of (6.30) vanishes in each case, giving

\[
\frac{d}{dt} \langle \hat{p} \rangle_\psi = -\langle \frac{dV}{dx} \rangle_\psi ,
\]
Since none of these operators is explicitly time-dependent, we have that $\frac{\partial \hat{H}}{\partial t} = \frac{\partial \hat{x}}{\partial t} = \frac{\partial \hat{p}}{\partial t} = 0$ and so the $\langle \frac{\partial A}{\partial t} \rangle_\psi$ term on the RHS of (6.30) vanishes in each case, giving

$$\frac{d}{dt} \langle \hat{p} \rangle_\psi = -\langle \frac{dV}{dx} \rangle_\psi,$$

$$\frac{d}{dt} \langle \hat{x} \rangle_\psi = \frac{1}{m} \langle \hat{p} \rangle_\psi,$$
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\begin{align*}
\frac{d}{dt} \langle \hat{p} \rangle_\psi &= -\langle \frac{dV}{dx} \rangle_\psi, \\
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\frac{d}{dt} \langle \hat{H} \rangle_\psi &= 0. 
\end{align*}
\tag{6.35}
\]
Applications of Ehrenfest’s theorem

Since none of these operators is explicitly time-dependent, we have that \( \frac{\partial \hat{H}}{\partial t} = \frac{\partial \hat{x}}{\partial t} = \frac{\partial \hat{p}}{\partial t} = 0 \) and so the \( \langle \frac{\partial A}{\partial t} \rangle_\psi \) term on the RHS of (6.30) vanishes in each case, giving

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These are quantum versions of the classical laws \( \frac{d}{dt} x = \frac{1}{m} p \) (which follows from \( p = mv \)).
Applications of Ehrenfest’s theorem

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Since none of these operators is explicitly time-dependent, we have that $\frac{\partial \hat{H}}{\partial t} = \frac{\partial \hat{x}}{\partial t} = \frac{\partial \hat{p}}{\partial t} = 0$ and so the $\langle \frac{\partial A}{\partial t} \rangle_\psi$ term on the RHS of (6.30) vanishes in each case, giving

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\[
\begin{align*}
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So the average behaviour predicted by quantum mechanics is consistent with classical mechanics for macroscopic systems. If that were not true, we should be able to detect discrepancies with classical mechanics, even for large objects, without doing complicated interference experiments.

For example, if the average energy for some quantum system was not conserved, we should be able to build an energy source or sink by making lots of copies of that system and letting it evolve.
\[
\frac{d}{dt} \langle \hat{p} \rangle_\psi = -\langle \frac{dV}{dx} \rangle_\psi,
\]
\[
\frac{d}{dt} \langle \hat{x} \rangle_\psi = \frac{1}{m} \langle \hat{p} \rangle_\psi,
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Note that Ehrenfest's theorem shows that expectation values follow equations analogous to classical laws, but does not describe the behaviour of uncertainties, which have no real classical analogue. For example, the uncertainty in position typically increases with time:

\[\Delta x\]
\[
\frac{d}{dt} \langle \hat{p} \rangle_{\psi} = -\langle \frac{dV}{dx} \rangle_{\psi}, \\
\frac{d}{dt} \langle \hat{x} \rangle_{\psi} = \frac{1}{m} \langle \hat{p} \rangle_{\psi}, \\
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These are quantum versions of the classical laws \( \frac{d}{dt} x = \frac{1}{m} p \) (which follows from \( p = mv \)), \( \frac{d}{dt} p = -\frac{dV}{dx} \) (which follows from \( F = ma \)), and \( \frac{d}{dt} E = 0 \) (conservation of total energy).

Consider the quantum harmonic oscillator: 

\[ V(x) = \frac{1}{2} m \omega^2 x^2 \]
\[ \frac{d}{dt} \langle x \rangle_\Psi = \frac{1}{i} \langle \hat{p} \rangle_\Psi \]
\[ \frac{d}{dt} \langle \hat{p} \rangle_\Psi = -\langle \frac{d\hat{V}}{dx} \rangle_\Psi \]
\[ = -i\omega^2 \langle \hat{x} \rangle_\Psi \]
\[ = -\frac{d^2}{dt^2} \langle \hat{x} \rangle_\Psi = -\omega^2 \langle \hat{x} \rangle_\Psi \]

\[ \langle \hat{x} \rangle_\Psi = A \cos \omega t + B \sin \omega t \]
\[ \langle \hat{p} \rangle_\Psi = A m \omega \sin \omega t + B m \omega \cos \omega t. \]

We get the same equations as those for \(x,p\) for the classical harmonic oscillator.

(Particular fact about the harmonic oscillator: not true for general potentials.)
For another interesting example where Ehrenfest's theorem leads to simple equations of motion for the expectation values, consider a linear potential $V(x) = Ax$. 

https://www.youtube.com/watch?v=1fMi1nriS8Q
Recall that the harmonic oscillator Hamiltonian is
Recall that the harmonic oscillator Hamiltonian is

\[ \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \]
The harmonic oscillator revisited

Recall that the harmonic oscillator hamiltonian is

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2
\]

\[
= \frac{1}{2m} (\hat{p} + i m \omega \hat{x})(\hat{p} - i m \omega \hat{x}) + \frac{i \omega}{2} [\hat{p}, \hat{x}]
\]
Recall that the harmonic oscillator hamiltonian is

\[ \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 \]

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\]

(6.36)

Define the operator \( a = \frac{1}{\sqrt{2m}} (\hat{p} - im\omega \hat{x}) \).
Recall that the harmonic oscillator hamiltonian is

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Recall that the harmonic oscillator hamiltonian is

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\[ \hat{H} = a^\dagger a + \frac{1}{2} \hbar \omega . \quad (6.37) \]
The harmonic oscillator revisited

We have the following commutation relations:

\[ [a, a^\dagger] = \frac{1}{2m} (-im\omega) 2[\hat{x}, \hat{p}] \]
The harmonic oscillator revisited

We have the following commutation relations:

\[ [a, a^\dagger] = \frac{1}{2m}(-im\omega)2[\hat{x}, \hat{p}] = \hbar \omega, \quad (6.38) \]
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Suppose now that \( \psi \) is a harmonic oscillator eigenfunction of energy \( E \):

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\hat{H}\psi = E\psi .
\]
**The harmonic oscillator revisited**

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We then have

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\hat{H}a\psi = [\hat{H}, a]\psi + a\hat{H}\psi
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\hat{H}a\psi = [\hat{H}, a]\psi + a\hat{H}\psi = (E - \hbar\omega)a\psi
\]

\underline{eigenvalue, new eigenfunction}
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\[
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*The harmonic oscillator revisited*

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\[
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\]

so that \( a\psi \) and \( a^\dagger\psi \) are eigenfunctions of energy \( (E - \hbar\omega) \) and \( (E + \hbar\omega) \).
We can use this to prove by induction that $a^n \psi$ and $(a^\dagger)^n \psi$ are eigenfunctions of energy $(E - n\hbar \omega)$ and $(E + n\hbar \omega)$. 
*The harmonic oscillator revisited

We can use this to prove by induction that $a^n\psi$ and $(a^\dagger)^n\psi$ are eigenfunctions of energy $(E - n\hbar\omega)$ and $(E + n\hbar\omega)$. For example,

$$\hat{H}a^n\psi = \hat{H}(a^{n-1}\psi)$$
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We can use this to prove by induction that $a^n\psi$ and $(a^\dagger)^n\psi$ are eigenfunctions of energy $(E - n\hbar\omega)$ and $(E + n\hbar\omega)$. For example,

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In particular, if it were true that $a^n\psi \neq 0$ for all $n$, 

The harmonic oscillator revisited

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In particular, if it were true that $a^n \psi \neq 0$ for all $n$, there would be eigenfunctions of arbitrarily low energy, and so there would be no ground state.
The harmonic oscillator revisited

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However, given any physical wavefunction $\psi$, we have that

$$\langle \hat{H} \rangle_\psi = \int_{-\infty}^{\infty} \psi^* \left( \frac{-\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi \right) dx$$
The harmonic oscillator revisited

We can use this to prove by induction that $a^n\psi$ and $(a^\dagger)^n\psi$ are eigenfunctions of energy $(E - n\hbar \omega)$ and $(E + n\hbar \omega)$. For example,

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$$\langle \hat{H} \rangle_\psi \overset{\text{def.}}{=} \int_{-\infty}^{\infty} \psi^*(\frac{-\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi)dx \geq 0,$$

$$\int_0^\infty (C \left| \frac{d\psi}{dx} \right|^2 + C' |\psi|^2)dx = \int_0^\infty \frac{\hbar^2}{2m} \frac{d\phi}{dx} \frac{d\phi}{dx} + \frac{1}{2}m\omega^2x^2\hat{r}(x)\hat{r}(x)\psi.$$

The harmonic oscillator revisited

We can use this to prove by induction that \( a^n \psi \) and \( (a^\dagger)^n \psi \) are eigenfunctions of energy \( (E - n\hbar\omega) \) and \( (E + n\hbar\omega) \). For example,

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\hat{H}a^n \psi = \hat{H}a(a^{n-1} \psi) = (E_{n-1} - \hbar\omega)a^n \psi,
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where \( E_r \) is the energy eigenvalue of \( a^r \psi \). Since \( E_0 = E \), it follows by induction that \( E_n = (E - n\hbar\omega) \).

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since both terms are non-negative.
The harmonic oscillator revisited

We can use this to prove by induction that $a^n\psi$ and $(a^\dagger)^n\psi$ are eigenfunctions of energy $(E - n\hbar\omega)$ and $(E + n\hbar\omega)$. For example,

$$\hat{H}a^n\psi = \hat{H}a(a^{n-1}\psi) = (E_{n-1} - \hbar\omega)a^n\psi,$$

(6.43)

where $E_r$ is the energy eigenvalue of $a^r\psi$. Since $E_0 = E$, it follows by induction that $E_n = (E - n\hbar\omega)$.

In particular, if it were true that $a^n\psi \neq 0$ for all $n$, there would be eigenfunctions of arbitrarily low energy, and so there would be no ground state.

However, given any physical wavefunction $\psi$, we have that

$$\langle \hat{H} \rangle_\psi = \int_{-\infty}^{\infty} \psi^* \left( \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \psi + \frac{1}{2} m\omega^2 x^2 \psi \right) dx \geq 0,$$

since both terms are non-negative.

So there cannot be negative energy eigenfunctions.
*The harmonic oscillator revisited

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Since \( \hat{H} = a^\dagger a + \frac{\hbar \omega}{2} \) and \( a\psi_0 = 0 \), we have \( \hat{H}\psi_0 = \frac{\hbar \omega}{2} \psi_0 \), giving us the previously obtained value of \( \frac{\hbar \omega}{2} \) for the ground state energy.
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ground state and hence for the excited states,

\[(a^\dagger)^n \psi_0 = C \left( \frac{1}{\sqrt{2m}}(\hat{p} + im\omega\hat{x}) \right)^n \exp\left(-\frac{m\omega x^2}{2\hbar}\right),\]  
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We have also obtained a closed form expression (6.46) for the ground state and hence for the excited states,

$$\left( a^\dagger \right)^n \psi_0 = C\left( \frac{1}{\sqrt{2m}}(\hat{p} + im\omega \hat{x}) \right)^n \exp\left( -\frac{m\omega x^2}{2\hbar} \right), \quad (6.47)$$

and we see immediately that their energies are \((n + \frac{1}{2})\hbar\omega\).

Note: we can also see that there cannot be eigenfunctions with energies other than these values. If there were, we could apply \((a)^n\) to them for arbitrarily large \(n\), without obtaining the zero function, and so there would be negative energy eigenstates.

With a little more thought we can also show that the eigenspaces must be nondegenerate.