$\lim_{\frac{V^+}{\text{step}}} \to V$

$V_+^+$ is a square well s.t. $V_+^+ = 0$ for $|x| \geq a$

$V_+^+(x) \not\geq V(x)$ (\forall x)$
There is at least one -ve energy bound state \( \psi \) for the SE with potential \( V_+ (x) \)

\[
\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_+ (x) \right) \psi_0 (x) = E \psi_0 (x)
\]

\( E < 0 \).

Suppose all energy eigenvalues for SE with potential \( V(x) \) are positive.

Then we can expand \( \psi_0 (x) \) as a superposition

\[
\sum_{i} c_i \psi_i (x) \quad \text{(or continuous sum)}
\]

of eigenstates \( \psi_i \) with eigenvalues \( E_i > 0 \).

Now \( 0 > E = \langle \hat{H} \rangle \psi_0 < 0 \) where \( \hat{A} = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \).

\[
\langle \hat{H} \rangle \psi_0 = \sum \psi_0 (x) \psi_0 (x) d x = \langle \hat{H} \rangle \psi_0
\]

where \( \hat{H} = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \).

However \( \langle \hat{H} \rangle \psi_0 = \sum c_i \psi_i (x) \quad \langle \hat{H} \rangle \sum c_i \psi_i (x) = \sum |c_i|^2 E_i > 0 \).
Spherically symmetric bound states of the hydrogen atom

We model the hydrogen atom by treating the proton as infinitely massive and at rest.

modelled by Coulomb potential

modelled as single particle wave function in Coulomb potential

\( m_p \sim 1836 \ m_e \) so this is a pretty good approximation
We model the hydrogen atom by treating the proton as infinitely massive and at rest. We seek spherically symmetric bound state wavefunctions $\psi(r)$ for the electron orbiting in a Coulomb potential $V(r) = -\frac{e^2}{4\pi\varepsilon_0 r}$.
Spherically symmetric bound states of the hydrogen atom

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\[
-\frac{\hbar^2}{2M} \left( \frac{d^2 \psi}{dr^2} + \frac{2}{r} \frac{d \psi}{dr} \right) - \frac{e^2}{4\pi\varepsilon_0 r} \psi(r) = E \psi(r),
\]

for some \( E < 0 \).
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$$\frac{d^2\psi}{dr^2} + \frac{2}{r} \frac{d\psi}{dr} + \frac{a}{r} \psi - b^2\psi = 0. \quad (7.25)$$
Spherically symmetric bound states of the hydrogen atom

\[ \frac{d^2 \psi}{dr^2} + \frac{2}{r} \frac{d \psi}{dr} + \frac{a}{r} \psi - b^2 \psi = 0. \]

Try the ansatz

\[ \psi(r) \approx \exp(-br), \]  \hspace{1cm} (7.26)
Spherically symmetric bound states of the hydrogen atom

\[ b^2 e^{-br} \]

\[ \frac{d^2}{dr^2} \frac{\psi}{r} + \frac{2}{r} \frac{d}{dr} \frac{\psi}{r} + \frac{a}{r} \psi - b^2 \psi = 0. \]

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The first and fourth terms dominate for large \( r \), and cancel one another precisely.
Spherically symmetric bound states of the hydrogen atom

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The first and fourth terms dominate for large \( r \), and cancel one another precisely. This suggests trying an ansatz of the form \( \psi(r) = f(r) \exp(-br) \), with \( f(r) = \sum_{n=0}^{\infty} a_n r^n \).
Spherically symmetric bound states of the hydrogen atom

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The first and fourth terms dominate for large \( r \), and cancel one another precisely. This suggests trying an ansatz of the form

\[ \psi(r) = f(r) \exp(-br), \quad \text{with} \quad f(r) = \sum_{n=0}^{\infty} a_n r^n, \text{in the hope of finding values of the coefficients } a_n \text{ such that the four terms cancel precisely to all orders.} \]
Our previous discussion assumed that $V(r)$ is nonsingular as $r \to 0$. 
Our previous discussion assumed that $V(r)$ is nonsingular as $r \to 0$. Here $V(r) \to \infty$ as $r \to 0$, so we cannot use the previous justification to argue that that $\phi(r) = r\psi(r) \to 0$ as $r \to 0$.

$$-rac{\hbar^2}{2M} \left( \frac{d^2 \psi}{dr^2} + \frac{2}{r} \frac{d \psi}{dr} \right) - \frac{e^2}{4\pi \epsilon_0 r} \psi(r) = E\psi(r), \quad (7.24)$$
Our previous discussion assumed that \( V(r) \) is nonsingular as \( r \to 0 \). Here \( V(r) \to \infty \) as \( r \to 0 \), so we cannot use the previous justification to argue that that \( \phi(r) = r\psi(r) \to 0 \) as \( r \to 0 \). However, we still require \( \psi(r) \) to define a normalisable 3D wavefunction, so that

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0 < \int_{r=0}^{\infty} r^2 |\psi(r)|^2 \, dr < \infty.
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\[
-\frac{\hbar^2}{2m} \nabla^2 \psi + V \psi = E \psi.
\]
Our previous discussion assumed that $V(r)$ is nonsingular as $r \to 0$. Here $V(r) \to \infty$ as $r \to 0$, so we cannot use the previous justification to argue that $\phi(r) = r \psi(r) \to 0$ as $r \to 0$. However, we still require $\psi(r)$ to define a normalisable 3D wavefunction, so that

$$0 < \int_{r=0}^{\infty} r^2 |\psi(r)|^2 \, dr < \infty.$$  

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I.e. \( \psi(r) = \left( \sum_{n=0}^{\infty} a_n r^n \right) e^{-b r} \) (not $n=-1$)
Spherically symmetric bound states of the hydrogen atom

We have

$$\frac{d^2 f}{dr^2} + \left( \frac{2}{r} - 2b \right) \frac{df}{dr} + \frac{1}{r} \left( a - 2b \right)f(r) = 0. \quad (7.27)$$

Try the ansatz

$$\frac{d^2 \psi}{dr^2} + \frac{2 d \psi}{r \, dr} + \frac{a}{r} \psi - b^2 \psi = 0.$$  

The first and fourth terms dominate for large $r$, and cancel one another precisely. This suggests trying an ansatz of the form

$$\psi(r) = f(r) \exp(-br), \quad \text{with} \quad f(r) = \sum_{n=0}^{\infty} a_n r^n,$$
Spherically symmetric bound states of the hydrogen atom

We have

\[ \frac{d^2 f}{dr^2} + \frac{2}{r} - 2b \frac{df}{dr} + \frac{1}{r} (a - 2b) f(r) = 0. \] (7.27)

Hence

\[ \sum_{n=0}^{\infty} (a_n n(n-1)r^{n-2} + 2a_n nr^{n-2} - 2ba_n nr^{n-1} + (a - 2b)a_n r^{n-1}) = 0 \]

(7.28)

Try the ansatz

\[ \psi(r) \approx \exp( - br ), \] (7.26)

The first and fourth terms dominate for large \( r \), and cancel one another precisely. This suggests trying an ansatz of the form

\[ \psi(r) = f(r) \exp( - br ), \] with

\[ f(r) = \sum_{n=0}^{\infty} a_n r^n, \]
We have
\[ \frac{d^2 f}{dr^2} + \left( \frac{2}{r} - 2b \right) \frac{df}{dr} + \frac{1}{r} (a - 2b) f(r) = 0. \] (7.27)

Hence
\[ \sum_{n=0}^{\infty} \left( a_n n(n-1) r^{n-2} + 2a_n nr^{n-2} - 2b a_n nr^{n-1} + (a - 2b) a_n r^{n-1} \right) = 0 \] (7.28)

Taking the coefficient of \( r^{n-2} \) we have
\[ a_n n(n-1) + 2a_n n - 2b a_{n-1} (n - 1) + (a - 2b) a_{n-1} = 0 \text{ for } n \geq 1. \] (7.29)
Spherically symmetric bound states of the hydrogen atom

This gives

\[ a_n = a_{n-1} \frac{(2b(n - 1) - (a - 2b))}{n(n - 1) + 2n} \]

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We thus have that \( a_n \to \frac{2b}{n} a_{n-1} \) for large \( n \).
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We thus have that \( a_n \to \frac{2b}{n} a_{n-1} \) for large \( n \).

If the coefficients do not vanish for large \( n \), this means they have the asymptotic behaviour of the coefficients of \( \exp(2br) \),

\[ e^{2br} = \sum \frac{1}{n!} (2br)^n = \sum c_n r^n \]

\[ \frac{c_n}{c_{n-1}} = \frac{2b}{n}. \]
This gives

\[ a_n = a_{n-1} \frac{(2b(n-1) - (a - 2b))}{n(n-1) + 2n} \]

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Spherically symmetric bound states of the hydrogen atom

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\( f(r) \approx C \exp(2br) \). This would give

\( \psi(r) \approx C \exp(2br) \exp(-br) \)
Spherically symmetric bound states of the hydrogen atom

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Spherically symmetric bound states of the hydrogen atom

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This gives

\[ a_n = a_{n-1} \frac{(2b(n - 1) - (a - 2b))}{n(n - 1) + 2n} \]

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If the coefficients do not vanish for large \( n \), this means they have the asymptotic behaviour of the coefficients of \( \exp(2br) \), i.e. \( f(r) \approx C \exp(2br) \). This would give \( \psi(r) \approx C \exp(2br) \exp(-br) = \exp(br) \), leading to an unnormalisable and thus unphysical wavefunction. So there must be some integer \( N \geq 1 \) for which \( a_N = 0 \), and we can take \( N \) to be the smallest such integer.
Then $a_{N-1} \neq 0$, so that $a_N = 0$ implies $2bN = a$ or $b = a/2N$.

\[
a_n = a_{n-1} \frac{(2b(n-1) - (a - 2b))}{n(n-1) + 2n} = a_{n-1} \frac{2bn - a}{n(n+1)}. \tag{7.30}
\]
Spherically symmetric bound states of the hydrogen atom

Then $a_{N-1} \neq 0$, so that $a_N = 0$ implies $2bN = a$ or $b = a/2N$, and so

$$E = -\frac{\hbar^2 a^2}{8MN^2}$$

$$a = \frac{e^2 M}{2\pi \varepsilon_0 \hbar^2}, \quad b = \frac{\sqrt{-2ME}}{\hbar}$$
Spherically symmetric bound states of the hydrogen atom

Then \( a_{N-1} \neq 0 \), so that \( a_N = 0 \) implies \( 2bN = a \) or \( b = a/2N \), and so

\[
E = -\frac{\hbar^2 a^2}{8MN^2} = -\frac{Me^4}{32\pi^2 \epsilon_0^2 \hbar^2 N^2},
\]

(7.31)

\[
a = \frac{e^2 M}{2\pi \epsilon_0 \hbar^2}, \quad b = \frac{\sqrt{-2ME}}{\hbar}
\]
Then \( a_{N-1} \neq 0 \), so that \( a_N = 0 \) implies \( 2bN = a \) or \( b = a/2N \), and so

\[
E = -\frac{\hbar^2 a^2}{8MN^2}
\]

\[
= -\frac{M e^4}{32\pi^2 \varepsilon_0^2 \hbar^2 N^2},
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which is precisely the energy spectrum of the Bohr orbits,
Then $a_{N-1} \neq 0$, so that $a_N = 0$ implies $2bN = a$ or $b = a/2N$, and so

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which is precisely the energy spectrum of the Bohr orbits, now derived from quantum mechanics, assuming spherical symmetry.
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which is precisely the energy spectrum of the Bohr orbits, now derived from quantum mechanics, assuming spherical symmetry. (We will need to drop this assumption to obtain the general orbital wavefunction).
Spherically symmetric bound states of the hydrogen atom

From

\[ a = 2bN \quad \text{and} \quad a_n = a_{n-1} \frac{2bn - a}{n(n + 1)} \]  

(7.32)
Spherically symmetric bound states of the hydrogen atom

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\[ a_n = a_{n-1} 2b \frac{n - N}{n(n+1)}. \quad (7.33) \]
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This gives solutions of the form

\[ f(r) = \begin{cases} 
1 & N = 1, \\
\end{cases} \]
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\[ f(r) = \begin{cases} 
1 & N = 1 , \\
(1 - br) & N = 2 , 
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This gives solutions of the form
\[ f(r) = \begin{cases} 
1 & N = 1, \\
(1 - br) & N = 2, \\
(1 - 2br + \frac{2}{3}(br)^2) & N = 3, 
\end{cases} \quad (7.34) \]
Spherically symmetric bound states of the hydrogen atom

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1 & N = 1, \\
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(1 - 2br + \frac{2}{3}(br)^2) & N = 3, 
\end{cases} \quad (7.34) \]

and generally \( f(r) = L^1_{N-1}(2br) \) where \( L^1_{N-1} \) is one of the associated Laguerre polynomials.
There is a simple expression for the associated Laguerre polynomials:
There is a simple expression for the associated Laguerre polynomials:

\[ L_N^k(x) = \frac{1}{N!} e^x x^{-k} \frac{d^N}{dx^N} (x^{N+k} e^{-x}). \]
Spherically symmetric bound states of the hydrogen atom

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Some plots of \( L_N^0 \) for small \( N \) and some other information about the Laguerre polynomials (\( L_N^k \) for \( k = 0 \)) and the associated Laguerre polynomials can be found at mathworld.wolfram.com/LaguerrePolynomial.html and at mathworld.wolfram.com/AssociatedLaguerrePolynomial.html.*
Spherically symmetric bound states of the hydrogen atom

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* The corresponding wavefunctions are \( \psi(r) = C L_{N-1}^1 (2br) \exp(-br) \), where the constant \( C \) is determined by normalisation.
From (7.2) we have

\[ \hat{x}_i = x_i \quad (\text{multiplication by } x_i), \quad (7.35) \]
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\[ \hat{p}_i = -i\hbar \frac{\partial}{\partial x_i}. \quad (7.36) \]
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\[ \hat{p}_i = -i\hbar \frac{\partial}{\partial x_i}. \] \hspace{1cm} (7.36)

By calculating the action on a general wavefunction as before, we obtain

\[ [\hat{x}_i, \hat{x}_j] = 0 = [\hat{p}_i, \hat{p}_j] \] \hspace{1cm} (7.37)
From (7.2) we have

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\[ [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}. \quad (7.38) \]
In classical mechanics we define the angular momentum vector

\[ L = x \wedge p, \]
Orbital Angular Momentum

In classical mechanics we define the angular momentum vector

\[ L = \mathbf{x} \wedge \mathbf{p}, \quad L_i = \epsilon_{ijk} x_j p_k, \quad (7.39) \]
In classical mechanics we define the angular momentum vector

\[
\mathbf{L} = \mathbf{x} \wedge \mathbf{p}, \quad L_i = \varepsilon_{ijk} x_j \hat{p}_k,
\]

(7.39)

\[
\hat{\mathbf{L}} = \hat{x}_2 \hat{p}_3 - \hat{x}_3 \hat{p}_2
\]

and \( \mathbf{L} \) is conserved in a spherically symmetric potential \( V(r) \).
In classical mechanics we define the angular momentum vector

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and \( L \) is conserved in a spherically symmetric potential \( V(r) \).

We define the quantum mechanical operators

\[ \hat{L} = -i\hbar \hat{x} \wedge \nabla, \]
Orbital Angular Momentum

In classical mechanics we define the angular momentum vector

\[ \mathbf{L} = \mathbf{x} \wedge \mathbf{p}, \quad L_i = \epsilon_{ijk} x_j p_k, \quad (7.39) \]

and \( \mathbf{L} \) is conserved in a spherically symmetric potential \( V(r) \).

We define the quantum mechanical operators

\[ \hat{\mathbf{L}} = -i\hbar \hat{\mathbf{x}} \wedge \nabla, \quad \hat{L}_i = -i\hbar \epsilon_{ijk} \hat{x}_j \frac{\partial}{\partial x_k}, \quad (7.40) \]
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(7.40)

and the total angular momentum

\[ \hat{\mathbf{L}}^2 = \hat{\mathbf{L}} \cdot \hat{\mathbf{L}} \]
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(7.40)

and the total angular momentum

\[ \hat{L}^2 = \hat{\mathbf{L}} \cdot \hat{\mathbf{L}} = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2. \]  

(7.41)
\[ [\hat{L}_i, \hat{L}_j] = -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} [\hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n \frac{\partial}{\partial x_p}] \]
\[ [\hat{L}_i, \hat{L}_j] = -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left[ \hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n \frac{\partial}{\partial x_p} \right] \]

\[ = -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( [\hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n] \frac{\partial}{\partial x_p} + \hat{x}_n [\hat{x}_l \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_p}] \right) \]
\[
[\hat{L}_i, \hat{L}_j] = -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left[ \hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n \frac{\partial}{\partial x_p} \right]
\]

\[
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( [\hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n] \frac{\partial}{\partial x_p} + \hat{x}_n [\hat{x}_l \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_p}] \right)
\]

\[
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( \hat{x}_l \left[ \frac{\partial}{\partial x_m}, \hat{x}_n \right] \frac{\partial}{\partial x_p} + \hat{x}_n \left[ \hat{x}_l, \frac{\partial}{\partial x_p} \right] \frac{\partial}{\partial x_m} \right)
\]
\[
\{ \hat{L}_i, \hat{L}_j \} = -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left[ \hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n \frac{\partial}{\partial x_p} \right]
\]

\[
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} (\left[ \hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n \right] \frac{\partial}{\partial x_p} + \hat{x}_n \left[ \hat{x}_l \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_p} \right])
\]

\[
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} (\hat{x}_l \left[ \frac{\partial}{\partial x_m}, \hat{x}_n \right] \frac{\partial}{\partial x_p} + \hat{x}_n \left[ \hat{x}_l, \frac{\partial}{\partial x_p} \right] \frac{\partial}{\partial x_m})
\]

\[
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} (\hat{x}_l \delta_{mn} \frac{\partial}{\partial x_p} - \hat{x}_n \delta_{lp} \frac{\partial}{\partial x_m})
\]
\[
[\hat{L}_i, \hat{L}_j] = -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left[ \hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n \frac{\partial}{\partial x_p} \right]
\]

\[
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( [\hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n] \frac{\partial}{\partial x_p} + \hat{x}_n [\hat{x}_l \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_p}] \right)
\]

\[
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( \hat{x}_l \left[ \frac{\partial}{\partial x_m} , \hat{x}_n \right] \frac{\partial}{\partial x_p} + \hat{x}_n \left[ \hat{x}_l , \frac{\partial}{\partial x_p} \right] \frac{\partial}{\partial x_m} \right)
\]

\[
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( \hat{x}_l \delta_{mn} \frac{\partial}{\partial x_p} - \hat{x}_n \delta_{lp} \frac{\partial}{\partial x_m} \right)
\]

\[
= -\hbar^2 \epsilon_{mil} \epsilon_{mpj} \hat{x}_l \frac{\partial}{\partial x_p} - \hbar^2 \epsilon_{pjn} \epsilon_{pmi} \hat{x}_n \frac{\partial}{\partial x_m}
\]
\[
[L_i, L_j] = -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left[ \hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n \frac{\partial}{\partial x_p} \right]
\]

\[
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( \left[ \hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n \right] \frac{\partial}{\partial x_p} + \hat{x}_n \left[ \hat{x}_l \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_p} \right] \right)
\]

\[
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( \hat{x}_l \left[ \frac{\partial}{\partial x_m}, \hat{x}_n \right] \frac{\partial}{\partial x_p} + \hat{x}_n \left[ \hat{x}_l, \frac{\partial}{\partial x_p} \right] \frac{\partial}{\partial x_m} \right)
\]

\[
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( \hat{x}_l \delta_{mn} \frac{\partial}{\partial x_p} - \hat{x}_n \delta_{lp} \frac{\partial}{\partial x_m} \right)
\]

\[
= -\hbar^2 \epsilon_{mil} \epsilon_{mpj} \hat{x}_l \frac{\partial}{\partial x_p} - \hbar^2 \epsilon_{pjn} \epsilon_{pmi} \hat{x}_n \frac{\partial}{\partial x_m}
\]

\[
= -\hbar^2 \left( \delta_{ip} \delta_{lj} - \delta_{ij} \delta_{lp} \right) \hat{x}_l \frac{\partial}{\partial x_p} - \left( \delta_{jm} \delta_{ni} - \delta_{ji} \delta_{nm} \right) \hat{x}_n \frac{\partial}{\partial x_m}
\]
\[
\begin{align*}
[\hat{L}_i, \hat{L}_j] &= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} [\hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n \frac{\partial}{\partial x_p}] \\
&= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} ([\hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n] \frac{\partial}{\partial x_p} + \hat{x}_n [\hat{x}_l \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_p}]) \\
&= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} (\hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n \frac{\partial}{\partial x_p} + \hat{x}_n [\hat{x}_l, \frac{\partial}{\partial x_p}] \frac{\partial}{\partial x_m}) \\
&= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} (\hat{x}_l \delta_{mn} \frac{\partial}{\partial x_p} - \hat{x}_n \delta_{lp} \frac{\partial}{\partial x_m}) \\
&= -\hbar^2 \epsilon_{mil} \epsilon_{mpj} \hat{x}_l \frac{\partial}{\partial x_p} - \hbar^2 \epsilon_{pjr} \epsilon_{pmi} \hat{x}_n \frac{\partial}{\partial x_m} \\
&= -\hbar^2 (\delta_{ip} \delta_{lj} - \delta_{ij} \delta_{lp}) \hat{x}_l \frac{\partial}{\partial x_p} - (\delta_{jm} \delta_{ni} - \delta_{ji} \delta_{nm}) (\hat{x}_n \frac{\partial}{\partial x_m}) \\
&= -\hbar^2 (\hat{x}_j \frac{\partial}{\partial x_i} - \delta_{ij} \hat{x}_l \frac{\partial}{\partial x_l} - \hat{x}_i \frac{\partial}{\partial x_j} + \delta_{ij} \hat{x}_l \frac{\partial}{\partial x_l})
\end{align*}
\]
\[
[\hat{L}_i, \hat{L}_j] = -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left[ \hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n \frac{\partial}{\partial x_p} \right] \\
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( [\hat{x}_l \frac{\partial}{\partial x_m}, \hat{x}_n] \frac{\partial}{\partial x_p} + \hat{x}_n [\hat{x}_l \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_p}] \right) \\
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( \hat{x}_l \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_p} + \hat{x}_n \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_m} \right) \\
= -\hbar^2 \epsilon_{ilm} \epsilon_{jnp} \left( \hat{x}_l \delta_{mn} \frac{\partial}{\partial x_p} - \hat{x}_n \delta_{lp} \frac{\partial}{\partial x_m} \right) \\
= -\hbar^2 \epsilon_{mil} \epsilon_{mpj} \hat{x}_l \frac{\partial}{\partial x_p} - \hbar^2 \epsilon_{pjn} \epsilon_{pmi} \hat{x}_n \frac{\partial}{\partial x_m} \\
= -\hbar^2 \left( \delta_{ip} \delta_{lj} - \delta_{ij} \delta_{lp} \right) \hat{x}_l \frac{\partial}{\partial x_p} - \left( \delta_{jm} \delta_{ni} - \delta_{ji} \delta_{nm} \right) \left( \hat{x}_n \frac{\partial}{\partial x_m} \right) \\
= -\hbar^2 \left( \hat{x}_j \frac{\partial}{\partial x_i} - \delta_{ij} \hat{x}_l \frac{\partial}{\partial x_l} - \hat{x}_i \frac{\partial}{\partial x_j} + \delta_{ji} \hat{x}_l \frac{\partial}{\partial x_l} \right) \\
= i\hbar \epsilon_{ijk} \hat{L}_k .
\]
\[ [\hat{L}^2, \hat{L}_i] = [\hat{L}_j \hat{L}_j, \hat{L}_i] \]
\[
[\hat{L}^2, \hat{L}_i] = [\hat{L}_j \hat{L}_j, \hat{L}_i] = [\hat{L}_j, \hat{L}_i] \hat{L}_j + \hat{L}_j [\hat{L}_j, \hat{L}_i]
\]
\[
[\hat{L}^2, \hat{L}_i] = [\hat{L}_j \hat{L}_j, \hat{L}_i] = [\hat{L}_j, \hat{L}_i] \hat{L}_j + \hat{L}_j [\hat{L}_j, \hat{L}_i] = i\hbar (\epsilon_{jik}(\hat{L}_k \hat{L}_j + \hat{L}_j \hat{L}_k))
\]
\[ [\hat{L}^2, \hat{L}_i] = [\hat{L}_j \hat{L}_j, \hat{L}_i] = [\hat{L}_j, \hat{L}_i] \hat{L}_j + \hat{L}_j [\hat{L}_j, \hat{L}_i] = i\hbar(\epsilon_{jik}(\hat{L}_k \hat{L}_j + \hat{L}_j \hat{L}_k)) = 0. \] (7.43)
\[
[\hat{L}^2, \hat{L}_i] = [\hat{L}_j \hat{L}_j, \hat{L}_i] \\
= [\hat{L}_j, \hat{L}_i] \hat{L}_j + \hat{L}_j [\hat{L}_j, \hat{L}_i] \\
= i\hbar (\epsilon_{jik} (\hat{L}_k \hat{L}_j + \hat{L}_j \hat{L}_k)) \\
= 0. 
\] (7.43)

Since the $\hat{L}_i$ do not commute, they are not simultaneously diagonalisable.
\[
[\hat{L}^2, \hat{L}_i] = [\hat{L}_j \hat{L}_j, \hat{L}_i] \\
= [\hat{L}_j, \hat{L}_i] \hat{L}_j + \hat{L}_j [\hat{L}_j, \hat{L}_i] \\
= i\hbar (\epsilon_{jik} (\hat{L}_k \hat{L}_j + \hat{L}_j \hat{L}_k)) \\
= 0. 
\] (7.43)

Since the $\hat{L}_i$ do not commute, they are not simultaneously diagonalisable. However, $\hat{L}^2$ and any one of the $\hat{L}_i$ can be simultaneously diagonalised, since $[\hat{L}^2, \hat{L}_i] = 0$. 

\[ 5 \quad \begin{cases} 
\hat{L}_3, \hat{L}_3 \neq 0 \\
\text{so not } \hat{L}_3 \text{ and } \hat{L}_2.
\end{cases} \]
We also have

\[
[\hat{L}_i, \hat{x}_j] = i\hbar \epsilon_{ijk} \hat{x}_k ,
\]  

(7.44)
Orbital Angular Momentum

We also have

\[
\begin{align*}
[\hat{L}_i, \hat{x}_j] &= i\hbar \epsilon_{ijk} \hat{x}_k, \\
[\hat{L}_i, \hat{p}_j] &= i\hbar \epsilon_{ijk} \hat{p}_k,
\end{align*}
\] (7.44) (7.45)
Orbital Angular Momentum

We also have

\[ [\hat{L}_i, \hat{x}_j] = i\hbar \epsilon_{ijk} \hat{x}_k, \]  \hspace{1cm} (7.44)

\[ [\hat{L}_i, \hat{p}_j] = i\hbar \epsilon_{ijk} \hat{p}_k, \]  \hspace{1cm} (7.45)

\[ [\hat{L}_i, \sum_j \hat{x}_j^2] = 2i\hbar \epsilon_{ijk} \hat{x}_j \hat{x}_k = 0, \]  \hspace{1cm} (7.46)
We also have

\begin{align*}
[\hat{L}_i, \hat{x}_j] &= i\hbar \epsilon_{ijk} \hat{x}_k , \\
[\hat{L}_i, \hat{p}_j] &= i\hbar \epsilon_{ijk} \hat{p}_k , \\
[\hat{L}_i, \sum_j \hat{x}_j^2] &= 2i\hbar \epsilon_{ijk} \hat{x}_j \hat{x}_k = 0 , \\
[\hat{L}_i, \sum_j \hat{p}_j^2] &= 2i\hbar \epsilon_{ijk} \hat{p}_j \hat{p}_k = 0 .
\end{align*}

(7.44) \quad (7.45) \quad (7.46) \quad (7.47)
Now we have that $\hat{r} = \sqrt{\sum_j \hat{x}_j^2}$. 
Orbital Angular Momentum

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Now we have that $\hat{r} = \sqrt{\sum_j \hat{x}_j^2}$. We also have that $[\hat{L}_i, \sum_j \hat{x}_j^2] = 0$. One can show from this that $[\hat{L}_i, \hat{r}] = 0$. More generally, one can show that $[\hat{L}_i, V(r)] = 0$ for any spherically symmetric potential $V(r)$. 