Exercise 1

1. Prove that if $A$ is positive semi-definite, then $A$ is Hermitian.

2. The action of a flip operator $F \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $\mathcal{H}_A, \mathcal{H}_B \simeq \mathbb{C}^d$, is defined through the following equation:

$$F |i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle,$$

where $\{|i\rangle\}_{i=1}^d$ denotes an orthonormal basis of $\mathbb{C}^d$. Write an explicit expression for $F$ in terms of the basis vectors $|i\rangle$. What are the eigenvalues of $F$ and what are their multiplicities? Express $F$ in terms of the MES

$$|\Omega\rangle := \frac{1}{\sqrt{d}} |i\rangle \otimes |i\rangle.$$

*Hint: Think of what mathematical operation you would use to relate one to the other.*

Use the relation that you obtain to rewrite equations (1.4) and (1.5) of Lemma 1 of Notes 7 (on the course webpage) using the flip operator instead of the MES.

Exercise 2 Show that any density operator of a qubit can be written as

$$\rho = \frac{1}{2} (I + \vec{s} \cdot \vec{\sigma}) = \frac{1}{2} \sum_{k=0}^3 s_k \sigma_k, \quad (1)$$

where $\vec{s} = (s_1, s_2, s_3)$ is a real three-dimensional vector such that $||\vec{s}|| \leq 1$. In the above $\vec{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$, with $\sigma_1, \sigma_2, \sigma_3$ being the three Pauli matrices $\sigma_x, \sigma_y, \sigma_z$. Moreover, $\sigma_0$ denotes the $2 \times 2$ identity matrix.
The vector $\vec{s}$ is usually referred to as the Bloch vector. The set of all Bloch vectors defines a sphere which is called the Bloch sphere. It provides a useful geometrical representation of the states of a qubit.

Show that $s_k = \text{Tr} \sigma_k \rho$. Show that $||\vec{s}|| = 1$ for a pure state $\rho$. What is the Bloch representation of the completely mixed state $\rho = I/2$? What do the North Pole and the South Pole of the Bloch sphere correspond to?

**Exercise 3** Given the density matrix

$$\rho = \begin{pmatrix}
\frac{3}{5} & \frac{1}{4} - i\frac{1}{6} \\
\frac{1}{4} + i\frac{1}{6} & \frac{2}{5}
\end{pmatrix}$$

what is the Bloch vector $\vec{s} = (s_x, s_y, s_z)$ for $\rho$? Is it a pure state or a mixed state? What is the probability that a measurement of the spin of the qubit along the Z-axis will yield a value +1?

**Exercise 4** Show that the set of density operators acting on a Hilbert space $\mathcal{H}$, where $\text{dim} \mathcal{H} = d$ is a convex subset of the real vector space of $d \times d$ Hermitian matrices. Show that pure states are extremal points of this set.

**Exercise 5** Find the Schmidt ranks for each of the following states

- $|\phi_1\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |44\rangle)$
- $|\phi_2\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$
- $|\phi_3\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$
- $|\phi_4\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |11\rangle)$

**Exercise 6** Let $|\Psi_{AR}\rangle$ and $|\Phi_{AR}\rangle$ be two purifications of a state $\rho$ of a system $A$ to a composite system $AR$. Prove that there exists a unitary transformation $U_R$ acting on system $R$ alone such that

$$|\Phi_{AR}\rangle = (I_A \otimes U_R)|\Psi_{AR}\rangle.$$ 

**Exercise 7 (Surjectivity of $\Lambda \mapsto J$)**

The Choi-Jamilkowski matrix $J \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$ and a linear map $\Lambda : \mathcal{M}_d \rightarrow \mathcal{M}_d$ are related by

$$J = (\Lambda \otimes \text{id}_d)(|\Omega\rangle \langle \Omega|)$$

$$\text{Tr}[A \Lambda(B)] = d \text{Tr}[J(A \otimes B^T)]$$
for any \( A \in \mathcal{M}_d \) and \( B \in \mathcal{M}_d \), where \(|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle |i\rangle\). Prove that the map \( \Lambda \mapsto J \) is surjective.

*Hint: Decompose \( J \) into a linear combination of rank-1 operators \(|\psi_i\rangle \langle \psi_j|\) and use \(|\psi_i\rangle = (R_i \otimes I) |\Omega\rangle\).*

**Exercise 8 (Unitary freedom in the Kraus decomposition)**

Suppose \( \{A_i\}_{i=1}^{n} \) and \( \{B_i\}_{i=1}^{n} \) are Kraus operators giving rise to quantum operations \( \Lambda \) and \( \Lambda' \), respectively. Prove that \( \Lambda = \Lambda' \) if and only if there exist complex numbers \( u_{ij} \) such that \( A_i = \sum_{j=1}^{n} u_{ij} B_j \), and \( u_{ij} \) are the elements of an \( n \times n \) unitary matrix.

**Exercise 9 (Number of Kraus operators)**

1. Consider a quantum operation \( \Lambda : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B) \). What is the maximal rank of its Choi state \( J(\Lambda) \)?

2. Prove that any quantum operation \( \Lambda \) on the state \( \rho \) of a system with a \( d \)-dimensional Hilbert space, there exists a Kraus decomposition with at most \( d^2 \) elements, i.e.,

\[
\Lambda(\rho) = \sum_{i=1}^{n} A_i \rho A_i^\dagger,
\]

where \( n \leq d^2 \).

**Exercise 10** Consider two quantum operations \( \Lambda_1 \) and \( \Lambda_2 \) acting on a single qubit, having Kraus representations

\[
\Lambda_1(\rho) = \sum_{k=1}^{2} A_k \rho A_k^\dagger; \quad \Lambda_2(\rho) = \sum_{k=1}^{2} V_k \rho V_k^\dagger,
\]

with

\[
A_1 = \frac{1}{\sqrt{2}} \sigma_0 ; A_2 = \frac{1}{\sqrt{2}} \sigma_z
\]

and

\[
V_1 = |0\rangle \langle 0| ; V_2 = |1\rangle \langle 1|.
\]

How do the actions of \( \Lambda_1 \) and \( \Lambda_2 \) differ from each other?
Exercise 11  Show that the following three operators form a POVM.

\begin{align*}
E_1 &= \frac{\sqrt{2}}{1 + \sqrt{2}} |1\rangle \langle 1 | \\
E_2 &= \frac{\sqrt{2}}{2 + 2\sqrt{2}}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|) \\
E_3 &= \mathbb{1} - E_1 - E_2
\end{align*}

Suppose Alice gives Bob a state prepared in one of the two states
\(|\psi_1\rangle = |0\rangle\) or \(|\psi_2\rangle = (|0\rangle + |1\rangle) / \sqrt{2}\). Show that if Bob does a measurement characterized by these POVM elements on the state he receives, he never makes an error of misidentification. Discuss the possible outcomes.