Quantum Information Theory

Lecture 17

1 Quantum Channels

1.1 Introduction

Quantum operations arise in various topics of Quantum Information Theory since they describe any physical process that a quantum-mechanical system undergoes, e.g., in time evolution of the state of an open system, quantum data compression etc. It also describes what happens to quantum information when it is transmitted from a sender (Alice) to a receiver (Bob) through a noisy quantum communications channel. The latter is analogous to the transmission of information through a classical communications channel. In view of this analogy, a quantum operation (or superoperator) is also referred to as a quantum channel. In this lecture we shall study some examples of quantum channels acting on a single qubit. There is a useful geometric representation of the state of a single qubit – namely, the Bloch (sphere) representation, which we shall use in our discussion of quantum channels. In this representation the state of a single qubit can be written as

$$\rho = \frac{1}{2}(I + \vec{s} \cdot \vec{\sigma})$$  (1)

where $\vec{s}$ is a 3-dimensional real vector such that $||\vec{s}|| \leq 1$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, with $\sigma_x, \sigma_y,$ and $\sigma_z$ being the Pauli matrices. This vector is known as the Bloch vector for the state $\rho$ (or alternatively, the spin polarization of the qubit).

A state $\rho$ is pure if and only if $||\vec{s}|| = 1$ (since in this case $\rho^2 = \rho$). Writing a pure state of a qubit as $|\Psi\rangle = a|0\rangle + b|1\rangle$, and setting $a = \cos \frac{\theta}{2}, b = e^{i\phi} \sin \frac{\theta}{2}$ we find that any pure state $\rho \equiv |\Psi\rangle\langle\Psi|$ of a single qubit can be viewed as a point $(\theta, \phi)$ on the surface of a unit sphere - the Bloch sphere. The corresponding Bloch vector is given by

$$\vec{s} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$$  (2)

Any mixed state corresponds to a point inside the Bloch sphere. If $\rho = \frac{I}{2}$ we have $\vec{s} = 0$. Hence the origin of the Bloch sphere corresponds to the completely mixed state. It is instructive to see how the Bloch sphere changes under the action of a single qubit channel.
Definition: (unital channel) A quantum channel Λ is said to be unital if Λ(I) = I.

We shall consider the following examples of single qubit channels.
1) Bit-flip channel 2) Phase-flip channel 3) Depolarizing channel and 4) Amplitude-damping channel.

1.2 Bit-flip channel

This channel flips the (spin of the) qubit with probability \( p \) and leaves it invariant with probability \( 1 - p \)

\[
\Lambda(\rho) = p\sigma_x \rho \sigma_x + (1 - p) \rho
\]  

(3)

The corresponding Kraus operators are

\[
A_1 = \sqrt{1 - p} I; \quad A_2 = \sqrt{p} \sigma_x
\]  

(4)

This channel leaves the \( x \)-axis of the Bloch sphere unchanged whereas it compresses the \( y \)- and \( z \)-axes by a factor of \( (1 - 2p) \). Hence, if \( \rho = (1/2)(I + \vec{s}, \vec{\sigma}) \) where \( \vec{s} = (s_x, s_y, s_z) \) then

\[
\Lambda(\rho) = \frac{1}{2} (I + \vec{s}, \vec{\sigma})
\]

with

\[
\vec{s} = (s_x, (1 - 2p)s_y, (1 - 2p)s_z).
\]

Consequently, states on the \( x \)-axis are left unchanged while states in the \( yz \)-plane are uniformly contracted by a factor \( (1 - 2p) \) [see e.g. p.376 Nielsen and Chuang].

Note that the above channels lie in the class of random unitary channels (also called mixing-enhancing channel). A channel in this class acts as follows:

\[
\Lambda(\rho) = \sum_{i} p_i U_i \rho U_i^\dagger,
\]  

(5)

where the \( U_i \) are unitary operators and \( \{p_i\} \) is a probability distribution. Why do you think it is called mixing-enhancing? Hint: majorization!

1.3 Phase flip channel

This channel leaves the qubit unchanged with probability \( 1 - p \) and causes a phase flip with a probability \( p \). Hence,

\[
\Lambda(\rho) = p\sigma_z \rho \sigma_z + (1 - p) \rho
\]  

(6)

The corresponding Kraus operators are \( A_1 = \sqrt{1-p} I, A_2 = \sqrt{p} \sigma_z \).

This leads to the compression of the Bloch sphere by a factor of \( 1 - 2p \) in the \( xy \)-plane. In the \( z \)-basis it suppresses the off-diagonal matrix elements of \( \rho \) by a factor of \( 1 - 2p \) while keeping diagonal ones unaltered.
1.4 Depolarizing channel

In this channel with probability \((1 - p)\) the qubit remains intact, while with probability \(p/3\) each of the following errors occur: bit flip \((\sigma_x)\), phase flip \((\sigma_z)\) and combined flip \((\sigma_y)\).
Hence, the overall probability of an error is \(p\).

\[
\Lambda(\rho) = (1 - p)\rho + \frac{p}{3}(\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z)
\]  \hspace{1cm} (7)

There are four Kraus operators:

\[
A_1 = \sqrt{1 - p}I \quad ; \quad A_2 = (\sqrt{p/3})\sigma_x \quad ; \quad A_3 = (\sqrt{p/3})\sigma_y \quad ; \quad A_4 = (\sqrt{p/3})\sigma_z
\]

The channel is unital (as are also the Bit flip- and Phase flip channels): \(\Lambda(I) = I\) since \(\sigma^2_\alpha = I\) for \(\alpha = x, y, z\).

It is easy to see that under the action of this channel, the Bloch vector (or the spin polarization vector) \(\vec{s}\) contracts to \(\vec{s}' = ((1 - 4p)/3)\vec{s}\) and so the channel is called depolarizing.
As a result, the Bloch sphere contracts uniformly by a factor \((1 - 4p)/3\) under the action of this channel.

A depolarizing channel has an alternative characterization. It is a channel which leaves a qubit unaffected with a certain probability say \((1 - q)\) and with probability \(q\) it replaces the state of the qubit with a completely mixed state \(I_2\). In other words the error completely randomizes the state with probability \(q\).

\[
\Lambda(\rho) = (1 - q)\rho + qI_2.
\]  \hspace{1cm} (8)

Note that for any state \(\rho\)

\[
\frac{I}{2} = (\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z)/4
\]

Hence

\[
\Lambda(\rho) = (1 - q)\rho + q\frac{I}{2}\]
\[
\quad = (1 - q)\rho + \frac{q}{4}\sum_{\alpha=x,y,z} \sigma_\alpha\rho\sigma_\alpha
\]  \hspace{1cm} (9)

Comparing (9) with (7) we find that \(q = (4/3)p\) which is less than one for \(p < 3/4\).

Note: The depolarizing channel can be generalized to quantum systems of dimension \(d > 2\)

\[
\Lambda(\rho) = q\frac{I}{d} + (1 - q)\rho;
\]  \hspace{1cm} (10)

since \(\frac{I}{d}\) represents the completely mixed state in a Hilbert space of \(d\)-dimensions.
1.5 Amplitude damping channel

This channel provides a simple model of the decay of an excited state of a 2-level atom due to spontaneous emission of a photon.

Let us start by considering the unitary evolution of the system (≡ the 2-level atom) and its environment. Then the channel $\Lambda$ can be obtained by taking a partial trace of the result of this evolution over the Hilbert space of the environment.

Let $|0\rangle_A$ denote the ground state of the 2-level atom $A$ and let $|1\rangle_A$ denote its excited state. Its environment is the electromagnetic field which is initially in the vacuum state $|0\rangle_E$ (i.e., ‘no photons’).

If the system is in a state $|1\rangle_A$ then there is a probability $p$ that it decays to the ground state $|0\rangle_A$ with the spontaneous emission of a photon. As a result of this, the environment makes a transition from the vacuum state $|0\rangle_E$ to a state with one photon denoted by $|1\rangle_E$

The evolution is described by a unitary transformation $U$ that acts on the atom and its environment according to

$$U : |0\rangle_A \otimes |0\rangle_E \mapsto |0\rangle_A \otimes |0\rangle_E$$  \hspace{1cm} (11)  

(i.e., If the atom is in the ground state and the environment is at zero temperature, then there is no transition) and

$$U : |1\rangle_A \otimes |0\rangle_E \mapsto \sqrt{1-p} |1\rangle_A |0\rangle_E + \sqrt{p} |0\rangle_A |1\rangle_E$$  \hspace{1cm} (12)  

Evaluating the partial trace over $\mathcal{H}_E$ (i.e., with respect to the basis vectors $\{|0\rangle_E, |1\rangle_E\}$, we obtain the Kraus operators

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

$$A_2 \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 0; \quad A_2 \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \sqrt{p} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \quad |1\rangle_A \mapsto |0\rangle_A$$

$$A_1 \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right); \quad A_1 \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \sqrt{1-p} \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$A_2$ describes the decay $|1\rangle_A \mapsto |0\rangle_A$ (quantum jump). $A_1$ describes how the state evolves if there is no decay and spontaneous emission.

The density matrix changes to

$$\rho \equiv \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \mapsto \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{pmatrix}$$

If $\gamma$ is the spontaneous decay rate per unit time, then in a small time interval $\Delta t$, decay occurs with probability $p = \gamma \Delta t << 1$. The state of the atom after a time $t = n\Delta t$ (for a
positive integer $n$) can be found by applying the channel $n$ times in succession, which we denote as $\Lambda^{(n)}$. The $\rho_{11}$ matrix element then decays as

$$\rho_{11} \mapsto (1 - p)^n \rho_{11} = (1 - \frac{\gamma t}{n})^n \rho_{11} \to e^{-\gamma t} \rho_{11} \quad \text{as } n \to \infty,$$

which is the expected exponential decay law. (Further discussion in lecture.) The off-diagonal entries of the state $\rho$ decay by a factor $(1 - p)^{2} = e^{-\frac{\gamma t}{2}}$ and hence after a time $t$ the state of the atom is given by

$$\rho(t) = \begin{pmatrix} \rho_{00} + (1 - e^{-\gamma t}) \rho_{11} & e^{-\frac{\gamma t}{2}} \rho_{01} \\ e^{-\frac{\gamma t}{2}} \rho_{10} & e^{-\gamma t} \rho_{11} \end{pmatrix}$$

Hence in the limit $t \to \infty$, the state of the atom is given by

$$\lim_{n \to \infty} \Lambda^{(n)}(\rho) \to \begin{pmatrix} \rho_{00} & + \rho_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(13)

Here the final state of the atom is its ground state. The amplitude damping channel is an example of a channel which takes a mixed initial state $\rho = \sum_{i,j=0}^{1} \rho_{ij} |i\rangle \langle j|$ to a pure final state $|0\rangle \langle 0|$. Note that the von Neumann entropy of the final state is zero, and is hence less than that of the initial state!!

The decrease of $S(\rho)$ should not be looked upon as information gain. This is because every mixed state decays to the ground state under the repeated actions of this channel and hence we lose the ability to distinguish between different possible preparations of the initial mixed state $\rho$.

**Note:** The amplitude damping channel is not unital

$$\Lambda(I) = \begin{pmatrix} 1 + p & 0 \\ 0 & 1 - p \end{pmatrix} \neq I$$

(14)