QUANTUM INFORMATION & COMPUTATION

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1 Postulate (QM4): Quantum measurements

An isolated (closed) quantum system has a *unitary* evolution. However, when an experiment is done to find out the properties of the system, there is an interaction between the system and the experimentalists and their equipment (i.e., the external physical world). So the system is no longer closed and its evolution is not necessarily unitary. The following postulate provides a means of describing the effects of a measurement on a quantum-mechanical system.

In classical physics the state of any given physical system can always in principle be fully determined by suitable measurements on a single copy of the system, while leaving the original state intact. In quantum theory the corresponding situation is bizarrely different – quantum measurements generally have only probabilistic outcomes, they are "invasive", generally unavoidably corrupting the input state, and they reveal only a rather small amount of information about the (now irrevocably corrupted) input state identity. Furthermore the (probabilistic) change of state in a quantum measurement is (unlike normal time evolution) not a unitary process. Here we outline the associated mathematical formalism, which is at least, easy to apply.

(QM4) Quantum measurements and the Born rule In Quantum Mechanics one measures an observable, i.e. a self-adjoint operator. Let A be an observable acting on the state space $\mathcal{V}$ of a quantum system Since A is self-adjoint, its eigenvalues are real. Let its *spectral projection* be given by $A = \sum_n a_n P_n$, where $\{a_n\}$ denote the set of eigenvalues of A and P_n denotes the orthogonal projection onto the subspace of $\mathcal{V}$ spanned by eigenvectors of A corresponding to the eigenvalue P_n. Note that if the eigenvalue a_n is non-degenerate, the projection P_n is of rank 1 $P_n = |\varphi_n\rangle\langle\varphi_n|$, where $|\varphi_n\rangle$ satisfies the eigenvalue equation: $A|\varphi_n\rangle = a_n|\varphi_n\rangle$. The outcome of a measurement of a $\mathbf{A}$ (i.e. the measured value) is an eigenvalue a_n (say) of A. If the system is in a state $|\psi\rangle$ just before the measurement, then the probability that the outcome is a_n is given by

$$p(a_n) = \langle\psi|P_n|\psi\rangle, \tag{1}$$

Moreover, as a result of this measurement, the state of the system becomes

$$\frac{P_n|\psi\rangle}{\langle\psi|P_n|\psi\rangle^{1/2}}. \tag{2}$$

This is hence the *post-measurement state* when the measurement outcome is a_n. Equations (1) and (2) constitute the so-called *Born Rule.*

This prescription tells us that if the measurement is repeated immediately, the measured value is again a_n, this time with probability one.

The above defined measurement is called a *von Neumann measurememnt* or *projective measurement* (the latter nomenclature arising from the fact that the Born Rule is given entirely in terms of *projection operators*). A more general definition of measurement (*generalized measurements or POVMs*) is used in the case of open quantum systems.

Note that measurement in Quantum Mechanics is *probabilistic*, i.e., Quantum Mechanics assigns probabilities to different possible outcomes of a measurement. Moreover, measurement *disturbs* the state of a system, taking it to an eigenstate of the measured observable. In particular, if A and B are two observables which do not commute, then a measurement of A will necessarily influence the outcome of a subsequent measurement of B. The fact that acquiring information about a quantum system inevitably disturbs the state of the system leads to important differences between Classical– and Quantum Information Theory.

1.1 Quantum measurement of a state relative to a basis

It is also useful (particularly for this course) to introduce the notion of the quantum measurement of a state $|\psi\rangle$ *relative to a given orthonormal basis of* $\mathcal{V}$.

Suppose we are given a (single copy of a) quantum state $|\psi\rangle$ of a quantum system with state space $\mathcal{V}$ of dimension n. Let $\mathcal{B} = \{|e_1\rangle, \ldots, |e_n\rangle\}$ be any orthonormal basis of $\mathcal{V}$ and write $|\psi\rangle = \sum_{i=1}^{n} a_i |e_i\rangle$. Then we can make a *quantum measurement of* $|\psi\rangle$ *relative to the basis* $\mathcal{B}$. This is sometimes called a (complete) von Neumann measurement or *projective measurement*. The possible outcomes are $j = 1, \ldots, n$ corresponding to the basis states $|e_j\rangle$. The probability of obtaining outcome j is

$$\begin{aligned} p(j) &= \langle\psi| P_j |\psi\rangle \quad \text{with } P_j = |e_j\rangle brae_j \\ &= |\langle e_j|\psi\rangle|^2 = |a_j|^2. \end{aligned} \tag{3}$$

Hence, the probability of the outcome j is given by the square of the modulus of the amplitude a_j. If outcome j is seen then after the measurement the state is no longer $|\psi\rangle$ but has been "collapsed" to $|\psi_{\text{after}}\rangle = |e_j\rangle$ i.e. the basis state corresponding to the seen outcome. Since $|\psi_{\text{after}}\rangle = |e_j\rangle$ corresponding to the seen outcome j, if we were to apply the measurement again we will simply see the *same* j with certainty, and not be able to sample the probability distribution $p_i = |a_i|^2$ again.

The qualifier "complete" in "complete projective measurement" refers to the fact that the projections here are into *one*-dimensional orthogonal subspaces (defined by the orthonormal basis). The notion of incomplete projective measurement generalises this to arbitrary decompositions of the state space into orthogonal subspaces (of arbitrary dimension, summing to the dimension of the full space).

Incomplete projective measurements
Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_d\}$ be any decomposition of the state space $\mathcal{V}$ into d mutually orthogonal subspaces i.e. $\mathcal{V}$ is the direct sum $\mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_d$. Let Π_i be the operation of projection into $\mathcal{E}_i$. Thus $\Pi_i \Pi_i = \Pi_i$ (property of any projection operator) and by orthogonality we have $\Pi_i \Pi_j = 0$ for all $i \neq j$. Then the *incomplete measurement of any state* $|\psi\rangle$

relative to the orthogonal decomposition $\{\mathcal{E}_1, \ldots, \mathcal{E}_d\}$ is the following quantum operation: the measurement outcomes are $i = 1, \ldots d$ and the probability of outcome i is:

$$p(i) = \langle\psi| \Pi_i |\psi\rangle$$

and the post-measurement state $|\psi_i\rangle$ for outcome i is the ("collapsed") projected vector renormalised to unit length:

$$|\psi_i\rangle = \Pi_i |\psi\rangle / \sqrt{p(i)}.$$

A complete projective measurement is thus clearly a special case in which all the subspaces have dimension one. Any incomplete measurement (with orthogonal decomposition $\{\mathcal{E}_1, \ldots, \mathcal{E}_d\}$) can be refined to a complete one by choosing an orthonormal basis of the state space that is consistent with the $\mathcal{E}_i$'s i.e. each $\mathcal{E}_i$ is spanned by a subset of the basis vectors. Then, by performing this complete measurement (instead of the incomplete one) we can recover the outcome probabilities of the incomplete measurement outcomes by summing all the probabilities corresponding to basis vectors in each subspace $\mathcal{E}_i$. However the post-measurement states will be different for the incomplete measurement and its refinement.

Remark: Note that the measurement of a quantum observable A, with spectral projection $A = \sum_j a_j P_j$ is the incomplete measurement relative to the orthogonal decomposition of the state space $\mathcal{V}$ of the system into eigenspaces of A, with the outcome being an eigenvalue a_j of A (rather than just j). If the eigenvalues are all non-degenerate, then the measurement is a complete projective measurement.

Example of an incomplete measurement. (Parity measurement). The parity of a 2-bit string $b_1 b_2$ is the mod 2 sum $b_1 \oplus b_2$. The parity measurement on two qubits is the incomplete measurement on the four dimensional state space with two outcomes (labelled 0 and 1), which on the computational basis states corresponds to the parity of the state label. Thus the corresponding orthogonal decomposition is $\mathcal{E}_0 = \text{span}\{|00\rangle, |11\rangle\}$ and $\mathcal{E}_1 = \text{span}\{|01\rangle, |10\rangle\}$. Upon measurement, the state $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ will give outcome 0 with probability $p_0 = |a|^2 + |d|^2$ and the post-measurement state would then be $|\psi_1\rangle = (a|00\rangle + d|11\rangle)/\sqrt{p_0}$. $\square$

Remark: For the purpose of obtaining information about the identity of the state of a quantum system, the actual choice of naming of the distinct outcomes is of no real consequence. We can just consider the probability Prob($j^{th} outcome$), where the j^{th} outcome could either be the label of a basis vector or the j^{th} eigenvalue of an observable. So, in this course we sometimes base our notion of a quantum measurement on the underlying orthogonal decomposition of the state space $\mathcal{V}$ of the system (being measured) rather than referring to particular observables. However, it is good to keep in mind that the physical observable is also of importance since the physical implementation of a measurement involves a physical interaction between the system and a "measuring apparatus" and if for example, the basis states $|0\rangle$ and $|1\rangle$ of a qubit being measured physically are spin-Z eigenstates or photon polarisations or two chosen energy energy levels in a Calcium atom (with corresponding quantum observables being spin, polarisation or energy respectively), this knowledge will have a crucial effect on how the measurement interaction for a standard basis measurement is actually implemented.

The Extended Born rule

We will often consider measurement of only some part of a composite system, which is in fact just a particular kind of incomplete measurement. The associated formalism for probabilities and post-measurement states is called the *Extended Born Rule* and we give an explicit description here (as it will be often used). Suppose $|\psi\rangle$ is a quantum state of a composite system S_1S_2 with state space $\mathcal{V} \otimes \mathcal{W}$, where $\mathcal{V}$ and $\mathcal{W}$ have dimensions m and n, respectively. Let $\mathcal{B} = \{|e_1\rangle, \ldots, |e_m\rangle\}$ be an orthonormal basis of $\mathcal{V}$ and $\mathcal{F} = \{|f_1\rangle, \ldots, |f_n\rangle\}$ be an orthonormal basis of $\mathcal{W}$. Then, as we know, $\{|e_i\rangle \otimes |f_j\rangle\}_{i=1,\ldots,m;j=1,\ldots,n}$ is an orthonormal basis of $\mathcal{V} \otimes \mathcal{W}$ and hence $|\psi\rangle$ can be expanded uniquely as

$$|\psi\rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} |e_i\rangle |f_j\rangle ,$$

with $a_{ij} \in \mathbb{C}$ and $\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 = 1$, since $|\psi\rangle$ is a state of a quantum system and is hence normalized.

Now we can make a measurement of $|\psi\rangle \in \mathcal{V} \otimes \mathcal{W}$ relative to the basis $\mathcal{B}$ of $\mathcal{V}$. This amounts to an incomplete measurement in $\mathcal{V} \otimes \mathcal{W}$ relative to its decomposition into mutually orthogonal subspaces $\mathcal{E}_i$ given by $\mathcal{E}_i = \text{span}\{|e_i\rangle \otimes |\phi\rangle \text{ for all } |\phi\rangle \in W\}$. The *Extended Born Rule* asserts the following:

(a) the probability of an outcome $k \in \{1, \ldots, m\}$ is given by

$$\begin{aligned} p(k) &= \langle\psi| (P_k \otimes I) |\psi\rangle , \quad \text{where} \quad P_k = |e_k\rangle \langle e_k| \\ &= \sum_{i,j,i',j'} a_{ij}^* \langle e_i| \langle f_j| (|e_k\rangle \langle e_k| \otimes I) a_{i'j'} |e_{i'}\rangle |f_{j'}\rangle \\ &= \sum_j |a_{kj}|^2 \end{aligned} \tag{4}$$

(b) if the outcome k is seen then the post-measurement state is

$$|\psi_{\text{after}}\rangle = \frac{(P_k \otimes I)}{\sqrt{p(k)}} |\psi\rangle = \frac{\sum_j a_{kj} |e_k\rangle |f_j\rangle}{\sqrt{\sum_{j'} |a_{kj'}|^2}}$$

Note: the basic Born rule is just a special case of (a) and (b) with $\mathcal{W}$ having dimension $n = 1$.

Remark: Fixed choice of basis:

Note that a measurement relative to any general basis $\mathcal{C}$ can be performed by a measurement relative to any a priori fixed basis $\mathcal{B}$ together with some unitary operations; indeed for any two orthonormal bases $\mathcal{B} = \{|e_1\rangle, \ldots, |e_m\rangle\}$ and $\mathcal{C} = \{|e'_1\rangle, \ldots, |e'_m\rangle\}$ of an m-dimensional state space $\mathcal{V}$, there is a unitary transformation U with $|e'_i\rangle = U|e_i\rangle$ for all i. Thus to perform a measurement on $|\psi\rangle \in \mathcal{V}$ relative to $\mathcal{C}$ we first apply U^{-1} to $|\psi\rangle$, then perform a measurement relative to $\mathcal{B}$, then finally apply U to the resulting post-measurement state, to obtain the same probabilities and post-measurement states as would have been obtained from a $\mathcal{C}$ measurement.

Standard measurement on multi-qubit systems

Recall that any n-qubit system comes equipped with a standard or computational basis

$\mathcal{B}$ of orthonormal states labelled by n-bit strings. In this course our measurements will often be restricted to being only relative to this standard basis for some subset of k qubits of an n-qubit system. We refer to such a measurement as a *standard measurement*.

Example: Consider the 3-qubit state $|\phi\rangle \in (\mathbb{C}^2)^{\otimes 3}$ given by

$$|\phi\rangle = \frac{i}{2}|000\rangle + \frac{1+i}{2\sqrt{2}}|001\rangle - \frac{1}{2}|101\rangle + \frac{3}{10}|110\rangle - \frac{2i}{5}|111\rangle .$$

A standard measurement of any of the three qubits state yields the outcome 0 or 1. The probability of the outcome 1 on making a standard measurement of the first qubit is, by the Extended Born Rule, given by

$$\begin{aligned} p(1) &= \langle\phi|(P_1 \otimes I \otimes I)|\phi\rangle , \quad \text{where } P_1 = |1\rangle\langle 1| \\ &= \frac{1}{4} + \frac{9}{100} + \frac{4}{25} \\ &= \frac{1}{2}. \end{aligned} \tag{5}$$

Compute the post-measurement state. □

Remark
According to (QM4), states with guaranteed different measurement outcomes always lie in orthogonal subspaces of the state space. Consequently two states are reliably physically distinguishable iff the corresponding kets are orthogonal. Here distinguishability means that there is a measurement which respectively outputs two distinct results, say 0 or 1, *with certainty* when applied respectively to the two states. We will explore consequences of this important non-classical feature much more later! – but we emphasise here that in contrast, in classical physics *any* two different states of a system are in principle distinguishable. □

Remark: global and relative phases revisited.
If $|v\rangle$ is any unit vector then the states $|v\rangle$ and $e^{i\theta}|v\rangle$ will have the same outcome probabilities (for a measurement relative to any basis or orthogonal decomposition), independent of θ (since probabilities always depend on squared moduli of amplitudes.) Also under unitary (hence linear) evolution the phase $e^{i\theta}$ just persists unchanged as a coefficient (i.e. a scalar multiplier). Here θ is called a *global phase*. Thus $|v\rangle$ and $e^{i\theta}|v\rangle$ represent identical physical situations and that is why in (QM1) we said that states of a physical system correspond to rays, i.e. to unit vectors *up to an (irrelevant) global phase*. Note also that the projection operator $\Pi_v = |v\rangle\langle v|$ is independent of the choice of global phase for $|v\rangle$ and hence it can also be used to uniquely represent distinct physical systems (not having the global phase ambiguity).
On the other hand θ in $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right)$ is called a *relative* phase and it is a crucially important parameter for the qubit state. Indeed for example, we can think of any unitary operation as evolving $|0\rangle$ and $|1\rangle$ separately and combining the results with relative phase θ which will affect the way that the two terms interfere (cf below). In the above, $\frac{1}{\sqrt{2}}$ is a normalization factor. A notable illustrative example is the pair of states $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. These differ only by a relative phase of π but they are easily seen to be orthogonal, so can be distinguished with certainty by a suitable measurement. □