# Distinguishing 2 mutually orthogonal states

1.2.20

Suppose $|\alpha_0\rangle$ & $|\alpha_1\rangle$ are 2 states in a state space $\mathcal{V}$ such that $\langle\alpha_0|\alpha_1\rangle = 0$, i.e. $|\alpha_0\rangle$ & $|\alpha_1\rangle$ are <u>mutually orthogonal</u>.

<u>Exercise</u>: Construct a projective measurement which can be used to distinguish between these 2 states with certainty

<u>SOLUTION</u>: Consider a 2 outcome measurement with projection operators $\Pi_0$ & $\Pi_1$ such that

$$\Pi_0 = |\alpha_0\rangle\langle\alpha_0| \; ; \; \Pi_1 = I - |\alpha_0\rangle\langle\alpha_0|$$

[<u>Note</u>: This is an <u>incomplete</u> projective measurement since $\Pi_1$ is not a rank-1 projector; $\Pi_0$ & $\Pi_1$ project onto mutually orthogonal subspaces of $\mathcal{V}$. Moreover, $\Pi_0|\alpha_0\rangle = |\alpha_0\rangle$; $\Pi_0|\alpha_1\rangle = 0$; $\Pi_1|\alpha_0\rangle = 0$, $\Pi_1|\alpha_1\rangle = |\alpha_1\rangle$ $\because \langle\alpha_0|\alpha_1\rangle = 0$]

Outcomes of this measurement: 0 ~~and~~ 1

Let $p(0)$ & $p(1)$ denote the corresponding probabilities.

- If the state is $|\alpha_0\rangle$,

$$p(0) = \langle\alpha_0|\Pi_0|\alpha_0\rangle = 1 \; ; \; p(1) = \langle\alpha_0|\Pi_1|\alpha_0\rangle = 0$$

whereas, if the state is $|\alpha_1\rangle$

$$p(0) = \langle\alpha_1|\Pi_0|\alpha_1\rangle = 0 \; ; \; p(1) = \langle\alpha_1|\Pi_1|\alpha_1\rangle = 1$$

- Hence using this measurement one can determine with certainty whether the state is $|\alpha_0\rangle$ or $|\alpha_1\rangle$:

Outcome $= 0 \Rightarrow$ state is $|\alpha_0\rangle$

Outcome $= 1 \Rightarrow$ state is $|\alpha_1\rangle$.