Distinguishing Bell states via Bell measurements

Suppose you are given a single copy of a state $|\Psi\rangle$ and told that it is one of the 4 Bell states.

How does a Bell measurement help you determine which state it is for sure? Doesn't the very first measurement that you do on $|\Psi\rangle$ cause a collapse?

No, it doesn't cause a collapse. This is the reason why:

Suppose the state $|\Psi\rangle$ is the Bell state $|\Psi_{AB}\rangle$.

Say you first ask the question: is the state $|\Psi_{AB}\rangle$?

To find the answer, you do a measurement corresponding to the projector $P_{00} = |\Phi^+\rangle\langle \Phi^+ |$.

Let $a$ denote the outcome of the measurement.

$\text{Prob}(a=00) = \langle \Psi | P_{00} | \Psi \rangle = \langle \Phi^+ | P_{00} | \Phi^+ \rangle = 0$

This means you do not get the outcome 00.

$\Rightarrow \text{Prob}(a \neq 00) = \text{Prob}(a \in \{01, 10, 11\}) = \langle \Phi^+ | (I - P_{00}) | \Phi^+ \rangle = 1$

"Post-measurement state $|\Psi\rangle \rightarrow (I - P_{00}) |\Psi\rangle = |\Phi^+ \rangle / \sqrt{1}"

So the state remains unchanged! There is no collapse.

Next ask: is the state $|\Phi^-\rangle$? Corresponding projector is $P_{01} = |\Phi^-\rangle\langle \Phi^- |$.

$\text{Prob}(a=01) = \langle \Psi^+ | P_{01} | \Psi^- \rangle = 0$

$\Rightarrow \text{Prob}(a \in \{10, 11\}) = \langle \Phi^+ | (I - P_{01}) | \Phi^+ \rangle = 1$

"Post-measurement state $|\Psi\rangle \rightarrow (I - P_{01}) |\Psi\rangle = |\Phi^- \rangle / \sqrt{1}"

Next ask: is the state $|\Psi^+\rangle$?; Projector $P_{10} = |\Psi^+\rangle\langle \Psi^+ |$.

$\text{Prob}(a=10) = \langle \Psi^+ | P_{10} | \Psi^+ \rangle = 1$

"Inference $|\Psi\rangle = |\Psi^+ \rangle$ with certainty"