Part III, Lent 2020
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Quantum Information Theory

Lecture 23

Superadditivity and superactivation of quantum capacity

In this lecture we will discuss one of the properties of quantum capacity which makes it notoriously hard to compute. We will see that there are channels which individually have zero quantum capacity, but when used jointly can have positive capacity. This suggests that quantum capacity depends on the context: on the availability of other resources.
One way to conclude that a channel has zero capacity is to show that bipartite states shared with the help of a channel between the sender and the receiver cannot be converted to a maximallly entangled state.

0.1 Local Operations and Classical Communication (LOCC)

LOCC operations represent the range of transformations which occur in a two (or more) spatially-separated laboratories where one is allowed to exchange classical messaages (without quantum communication). We will consider the case of two laboratories, with Alice representing the first laboratory and Bob the second. LOCC transformations consist of the following sequence of actions:

1. Alice performs a quantum operation (represented by a quantum channel) which may have both classical and quantum outputs. She then communicates her classical information to Bob

2. Bob performs a quantum operation which is conditioned on the classical information he has received from Alice. This can be mathematically described as $\sum_x \Lambda_A^{(x)} \otimes N_B^{(x)}$, where $\{\Lambda^{(x)}\}_x$ is a collection of CP maps such that $\sum_x \Lambda^{(x)}$ is a quantum channel, and each $N_B^{(x)}$ represents a quantum channel (for each x), and x denotes classical message.

3. Bob and Alice swap places and Bob performs actions in point 1) and Alice in point 2).

We will denote LOCC$^{\rightarrow}$ if we only allow unidirectional classical communication from the sender to the receiver). Similarly, we define LOCC$^{\leftarrow}$ if we only allow backwards classical comunication and LOCC$^{\leftrightarrow}$ if we allow unrestricted bidirectional communication.

0.2 Superactivation

Definition 1 *The state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is called distillable if $\exists n$ such that $\rho_{AB}^{\otimes n}$ can be locally projected onto an entangled state: $\exists P_A, Q_B$ - 2-dimensional projectors on systems A and B respectively $\left[(P_A \otimes Q_B)\rho_{AB}^{\otimes n}(P_A \otimes Q_B)\right]^{T_A}$ has at least one negative eigenvalue.*

We say that the state ρ_{AB} is LOCC-distillable if there is a sequence of LOCC operations Λ: $\Lambda(\rho_{AB}^{\otimes n}) = (\Phi_{AB}^{+})^{\otimes m}, m \geq 1$.

Lemma 1 *ρ_{AB} is distillable iff $\exists n$: $\rho_{AB}^{\otimes n} \to (LOCC)(\Phi_{AB}^{+})^{\otimes m}, m \geq 1$.*
<u>Alternative statement:</u> ρ_{AB} is distillable iff there exists a bipartite pure state $|\Psi\rangle$ of Schmidt rank 2 such that $\langle\Psi|(\rho_{AB}^{\otimes n})^{\Gamma_A}|\Psi\rangle < 0$.

Proof of the above lemma is non-examinable. The above lemma states that a state with a positive partial transpose (PPT) is non-distillable. Therefore, if we construct a channel which on any input produces a PPT state, then we can conclude that such channel has zero quantum capacity. We will present such a channel by constructing its Choi state.

Consider the state

$$\rho_{ABA'B'} = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+|_{AB} \otimes \tau_{1,A'B'} + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-|_{AB} \otimes \tau_{2,A'B'}, \tag{1}$$

where $\tau_{1,A'B'} = [(\rho_s + \rho_a)/2]^{\otimes k}$, $\tau_{1,A'B'} = [\rho_s]^{\otimes k}$, with $\rho_s = 2/(d^2 + d)P_{sym}$, $\rho_a = 2/(d^2 - d)P_{asym}$, where the latter are projectors on symmetric and antisymmetric subspaces respectively. From example sheets we know that $P_{sym} = 1/2(1 + \mathbb{F}), P_{asym} = 1/2(1 - \mathbb{F})$, with $\mathbb{F}$ being the permutation operator. This state has a special significance because it may be used to 'hide' entanglement (under LOCC operations).

Exercise [Optional and non-examinable]
Show that:

1. $\tau_{1,A'B'}, \tau_{2,A'B'}$ are separable.
2. $\mathrm{Tr}_{A'}\tau_{i,A'B'} = \mathrm{Tr}_{B'}\tau_{i,A'B'}$
3. $||\tau_1 - \tau_2|| \geq 0$, hence they are 'globally distinguishable'.

The state $\rho_{ABA'B'}$ has a negative partial transpose. This can be directly verified by inspecting $\rho_{ABA'B'}^{T_{BB'}}$. To make $\rho_{ABA'B'}$ have PPT, we will add a 'noise' term:

$$\rho_{(p,d,k)} = 2p\rho_{ABA'B'} + (1 - 2p)\frac{1}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes \tau_{2,A'B'}, \tag{2}$$

where $p \in (0, \frac{1}{2})$.
The following lemma provides a range for which $\rho_{(p,d,k)}$ is PPT:

Lemma 2 *For $p \in (0, \frac{1}{3})$, $\frac{1-p}{p} \geq \left(\frac{d}{d-1}\right)^k$, the state $\rho_{(p,d,k)}$ has PPT.*

Proof of this statement is located on page 32 of Appendix XV-B in arXiv:quant-ph/0506189. Proof is non-examinable.
Consider a quantum channel $\Lambda_{(p,d,k)}$ whose Choi state is $\rho_{(p,d,k)}$ with parameters satisfying conditions of Lemma 2. Thus its quantum capacity is $\mathcal{Q}(\Lambda_{(p,d,k)}) = 0$.

Another channel which we briefly considered is the Quantum Erasure channel:

$$\Lambda_p(\rho) = (1-p)\rho + p|e\rangle\langle e|, \tag{3}$$

where $\langle e|\rho|e\rangle = 0$ for all input states ρ. To see why $\Lambda_{\frac{1}{2}}(\rho)$ has zero quantum capacity, suppose there is a scheme consisting of an encoder and decoder which allows Alice to communicate quantum information to Bob reliably at a positive rate. Looking at the isometric extension of $\Lambda_{\frac{1}{2}}$ we see that Bob and the environment are treated identically:

$$\mathrm{id}_R \otimes \Lambda_{\frac{1}{2}}(|\psi\rangle\langle\psi|_{RA}|) = \mathrm{Tr}_E(\sqrt{1/2}|\psi\rangle\langle\psi|_{RB}| \otimes |e\rangle\langle e|_E + \sqrt{1/2}|\psi\rangle\langle\psi|_{RE}| \otimes |e\rangle\langle e|_B). \tag{4}$$

Then the environment could resort to using the same decoder that is used by Bob, which would lead to violation of no-cloning theorem. Thus $\mathcal{Q}(\Lambda_{\frac{1}{2}}) = 0$.
Counterintuitively, when they are used jointly we get $\mathcal{Q}(\Lambda_{(p,d,k)} \otimes \Lambda_{\frac{1}{2}}) > 0$!
Below is the sequence of actions which would lead to superactivation phenomenon:

1. Alice starts with $\Phi^+_{AB} \otimes \Phi^+_{A'B'}$ and feeds BB' into $\Lambda_{(p,d,k)} : \mathcal{D}(\mathcal{H}_{BB'}) \to \mathcal{D}(\mathcal{H}_{BB'})$.

2. Alice and Bob now share $\rho_{(p,d,k),ABA'B'}$, where states in A' and B' are separable.

3. Alice sends A' through the $\Lambda_{\frac{1}{2}}$. Half of the time (when erasure doesn't take place) Bob is able to distinguish between τ_1 and τ_2 and thus with non-zero probability q (which only depends on the admixed noise) learn which of the maximally entangled states ($\Phi^{\pm}$) they share. Thus, $Q(\Lambda_{(p,d,k)} \otimes \Lambda_{\frac{1}{2}}) \geq q/2 * I_c(A\rangle B) > 0$!

The magnitude of this effect can be made large $O(\log(\dim(\mathcal{H}_A)))$.